ON MIXTURE REPRESENTATION OF THE LINNIK DENSITY

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Abstract

Let $p_{\alpha,\theta}$ be the *Linnik density*, that is, the probability density with the characteristic function

$$\varphi_{\alpha,\theta}(t) := 1/(1 + e^{i\theta \operatorname{sgn} t}|t|^{\alpha}), \qquad (\alpha, \theta) \in PD,$$
$$PD := \{(\alpha, \theta) : 0 < \alpha < 2, |\theta| \le \min(\pi\alpha/2, \pi - \pi\alpha/2)\}.$$

The following problem is studied: Let (α, θ), (β, ϑ) be two points of PD. When is it possible to represent $p_{\beta,\vartheta}$ as a scale mixture of $p_{\alpha,\theta}$? A subset of the admissible pairs (α, θ), (β, ϑ) is described.

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1. Introduction and statement of results

In 1953, Linnik [12] considered a family $\{p_\alpha(x) : \alpha \in (0, 2)\}$ of symmetric probability densities with the characteristic functions

$$\varphi_\alpha(t) = 1/(1 + |t|^\alpha), \qquad 0 < \alpha < 2.$$

Since then, the family had several probabilistic applications ([1–5]). Analytic and asymptotic properties of the densities p_α were studied in [9].

We will consider a more general family $\{p_{\alpha,\theta}(x)\}$ of densities with characteristic functions

$$(1) \qquad \varphi_{\alpha,\theta}(t) = 1/(1 + e^{i\theta \operatorname{sgn} t}|t|^{\alpha}),$$
$$(\alpha, \theta) \in PD := \{(\alpha, \theta) : \alpha \in (0, 2), |\theta| \le \min(\pi\alpha/2, \pi - \pi\alpha/2)\}.$$

We will call the densities $p_{\alpha,\theta}$ the *Linnik densities*. Comparison of (1) with the well-known representation of a stable characteristic function (see, for example, [19, p. 17]) shows that the $p_{\alpha,\theta}$'s are exponential mixtures of stable densities. Evidently, $\varphi_{\alpha,0} = \varphi_\alpha$, $p_{\alpha,0} = p_\alpha$ and, moreover, $p_{\alpha,\theta}$ is non-symmetric for $\theta \neq 0$. For $|\theta| = \min(\pi\alpha/2, \pi - \pi\alpha/2)$ these densities first appeared in the paper of Laha [11]. Klebanov et al. [7] introduced the concept of geometric strict stability and proved that the family of the Linnik densities coincides with the family of geometrically strictly stable densities. Pakes [15–18] showed that the densities $p_{\alpha,\theta}$ play an important role in some characterization problems of mathematical statistics. Analytic and asymptotic properties of $p_{\alpha,\theta}$ were studied in [6, 10].

Kotz and Ostrovskii [8] proved that, if $0 < \alpha < \beta \leq 2$, then p_α can be represented as a scale mixture of p_β. This paper is devoted to a generalization of the result to the whole family of Linnik densities. Since a general expression of the Linnik densities is not easily attainable, such a mixture representation which facilitates generation of Linnik's densities seems to be of interest.

The problem studied in this paper is the following. Let (α, θ), (β, ϑ) be two points of PD. When is it possible to represent $\varphi_{\beta,\vartheta}$ as a scale mixture of $\varphi_{\alpha,\theta}$?

To state the result, let us denote by $PD_{\alpha,\theta}$ the subset $\{(\beta, \vartheta) \in PD : \beta \leq \alpha, |\vartheta|/\beta \leq |\theta|/\alpha\} \backslash \{(\alpha, \theta), (\alpha, -\theta)\}$ of PD (see Figure 1).

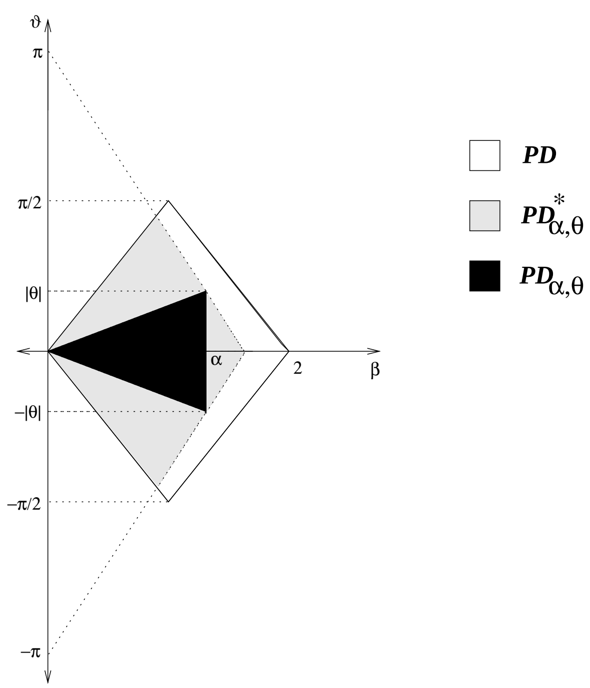

FIGURE 1

THEOREM 1. *If* $\theta \neq 0$, $(\alpha, \theta) \in PD$, *then for any* $(\beta, \vartheta) \in PD_{\alpha,\theta}$ *the following representation is valid:*

$$\varphi_{\beta,\vartheta}(t) = \int_{-\infty}^{\infty} \varphi_{\alpha,\theta}(t/s) g(s; \alpha, \beta, \theta, \vartheta)\, ds, \tag{2}$$

where $g(s; \alpha, \beta, \theta, \vartheta)$ *is a probability density.*

In virtue of Theorem 1, we obtain the representation

$$p_{\beta,\vartheta}(x) = \int_{-\infty}^{\infty} p_{\alpha,\theta}(xs) g(s; \alpha, \beta, \theta, \vartheta) s\, ds$$

as stipulated in the abstract.

We could not determine the maximal subset $PD^{+}_{\alpha,\theta}$ of PD, where $\varphi_{\beta,\vartheta}$ is a mixture of $\varphi_{\alpha,\theta}$ of the form (2). Nevertheless, the representation (2) remains valid for a larger set of (β, ϑ) if we do not require that $g(s; \alpha, \beta, \theta, \vartheta)$ is a probability density.

Denote by $PD^{\star}_{\alpha,\theta}$ the subset $\{(\beta, \vartheta) \in PD : \pi/\beta + |\theta|/\alpha > \pi/\alpha + |\vartheta|/\beta\}$ of PD (see Figure 1). Evidently $PD_{\alpha,\theta}$ is a proper subset of $PD^{\star}_{\alpha,\theta}$.

THEOREM 2. *If* $\theta \neq 0$, $(\alpha, \theta) \in PD$, *then for any* $(\beta, \vartheta) \in PD^{\star}_{\alpha,\theta}$ *the representation* (2) *is valid with*

$$g(\pm s; \alpha, \beta, \theta, \vartheta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^{\pm}(z; \alpha, \beta, \theta, \vartheta) s^{-z-1}\, dz, \qquad s > 0, \tag{3}$$

$$|c| < \min(\beta, \alpha\pi/(2|\theta|)),\ f^{\pm}(z; \alpha, \beta, \theta, \vartheta) = \frac{\alpha}{\beta} \frac{(\sin \pi z/\alpha)}{(\sin \pi z/\beta)} \frac{\sin z(\theta/\alpha \pm \vartheta/\beta)}{\sin(2z\theta/\alpha)}. \tag{4}$$

Theorem 1 is an immediate corollary of Theorem 2 and the following one.

THEOREM 3. *For any* $(\alpha, \theta) \in PD$ *such that* $\theta \neq 0$, *and any* $(\beta, \vartheta) \in PD_{\alpha,\theta}$, *the function* $g(s; \alpha, \beta, \theta, \vartheta)$ *defined by* (3) *is a probability density function.*

In connection with the open question about the size of the set $PD^{+}_{\alpha,\theta}$, it is of some interest that the function $g(s; \alpha, \beta, \theta, \vartheta)$ is not a probability density for $(\beta, \vartheta) \in PD^{\star}_{\alpha,\theta}$ (see Figure 1) lying to the right of the line $\{(\beta, \vartheta) : \beta = \alpha\}$ unless $(\alpha, \theta) = (1, \pm\pi/2)$ as the following remark shows.

REMARK. If $(\alpha, \theta) \notin \{(1, \pi/2), (1, -\pi/2)\}$, and $(\beta, \vartheta) \in (PD^{\star}_{\alpha,\theta} \setminus PD_{\alpha,\theta}) \bigcap \{(\beta, \vartheta) : \beta > \alpha\}$, then $g(s; \alpha, \beta, \theta, \vartheta)$ admits negative values and therefore is not a probability density.

The case $(\alpha, \theta) = (1, \pm\pi/2)$ is exceptional as we will see later (Theorem 5).

In [8], it was shown that for $0 < \beta < \alpha < 2$, $\varphi_\beta(t) = \int_0^\infty \varphi_\alpha(t/s) g(s, \alpha, \beta)\, ds$ where

$$g(s, \alpha, \beta) = \frac{\alpha}{\pi} \sin \frac{\pi\beta}{\alpha} \frac{s^{\beta-1}}{1 + s^{2\beta} + 2s^\beta \cos \pi\beta/\alpha}, \qquad s > 0. \tag{5}$$

This result is a limiting case of Theorem 3 since the following formula is valid for $\theta/\alpha = \vartheta/\beta$:

$$\lim_{\theta \to +0} g(s; \alpha, \beta, \theta, \vartheta) = \frac{1 + \operatorname{sgn} s}{2} g(|s|, \alpha, \beta).$$

Under the conditions of Theorem 3 the probability density $g(s; \alpha, \beta, \theta, \vartheta)$ is not concentrated on $\mathbb{R}^+$ in general. Before giving a description of its structure, we note that from (3) it follows that $g(s; \alpha, \beta, \theta, \vartheta) = g(s; \alpha, \beta, -\theta, -\vartheta)$, $g(s; \alpha, \beta, \theta, \vartheta) = g(-s; \alpha, \beta, \theta, -\vartheta)$. Therefore we can restrict our attention to the case when both θ and ϑ are positive.

Recall that the Mellin convolution of two functions $g_1, g_2 \in L(\mathbb{R}^+)$ is defined by the formula

$$(g_1 \star g_2)(x) = \int_0^\infty g_1(x/s) g_2(s)\, \frac{ds}{s}.$$

THEOREM 4. *Assume the conditions of Theorem 3 are satisfied.*

(i) *If $\beta < \alpha$, $\vartheta/\beta = \theta/\alpha$, then $g(s; \alpha, \beta, \theta, \vartheta)$ is concentrated on $\mathbb{R}^+$ and has the form*

$$g(s; \alpha, \beta, \theta, \vartheta) = \frac{1 + \operatorname{sgn} s}{2} g(|s|, \alpha, \beta). \tag{6}$$

(ii) *If $\beta = \alpha$, $0 < \vartheta < \theta$, then*

$$g(\pm s; \alpha, \beta, \theta, \vartheta) = \frac{\theta \pm \vartheta}{2\theta} g\left(s, \frac{\pi\alpha}{\theta \pm \vartheta}, \frac{\pi\alpha}{2\theta}\right), \qquad s > 0. \tag{7}$$

(iii) *In other cases*

(8)
$$g(\pm s; \alpha, \beta, \theta, \vartheta) = \frac{\theta\beta \pm \alpha\vartheta}{2\theta\beta}\left(g(s, \alpha, \beta) \star g\left(s, \frac{\pi\alpha\beta}{\theta\beta \pm \vartheta\alpha}, \frac{\pi\alpha}{2\theta}\right)\right), \qquad s > 0.$$

THEOREM 5. *For any $(\beta, \vartheta) \in PD \setminus \{(1, \pi/2), (1, -\pi/2)\}$ the following representation is valid.*

$$\varphi_{\beta,\vartheta}(t) = \int_{-\infty}^\infty \varphi_{1,\pi/2}(t/s) q(s; \beta, \vartheta)\, ds = \int_{-\infty}^\infty \frac{s}{s + it} q(s; \beta, \vartheta)\, ds \tag{9}$$

where q is a probability density given by the formula

$$q(\pm s;\beta,\vartheta)=\frac{\pi\beta\pm 2\vartheta}{2\pi\beta}g\left(s,\frac{2\pi\beta}{\pi\beta\pm 2\vartheta},\beta\right),\qquad s>0. \tag{10}$$

The representation (9) shows that all Linnik densities are mixtures of standard exponential densities $p_{1,\pm\pi/2}$.

2. Proof of the theorems

PROOF OF THEOREM 2. For simplicity, we shall write $f^{\pm}(z)$ instead of $f^{\pm}(z;\alpha,\beta,\theta,\vartheta)$. From (4) it follows that both functions $f^{+}(z)$ and $f^{-}(z)$ are analytic outside of the set

$$\left\{\{q\beta\}_{q=-\infty}^{\infty}\bigcup\{\pi\alpha q/(2\theta)\}_{q=-\infty}^{\infty}\right\}\setminus\{0\}.$$

Moreover, in any set $\{z:|\operatorname{Re}z|<H,|\operatorname{Im}z|>\varepsilon\}$, the following bound holds

$$|f^{\pm}(z)|\le C\exp(-D|\operatorname{Im}z|) \tag{11}$$

where C, D are positive constants not depending on z. Since $f^{\pm}(z)$ is analytic in $\{z:|\operatorname{Re}z|<\min(\beta,\pi\alpha/(2|\theta|))$, the integral in (3) does not depend on c under the restrictions mentioned in (4).

Denote by $I(t)$ the integral in the right hand side of (2). We show that it is equal to $\varphi_{\beta,\vartheta}(t)$.

Assume $t>0$. We have

$$\begin{aligned}I(t)&:=\int_{-\infty}^{\infty}\varphi_{\alpha,\theta}(t/s)g(s;\alpha,\beta,\theta,\vartheta)\,ds\\&=\left(\int_0^1+\int_1^{\infty}\right)\varphi_{\alpha,\theta}(-t/s)g(-s;\alpha,\beta,\theta,\vartheta)\,ds\\&\quad+\left(\int_0^1+\int_1^{\infty}\right)\varphi_{\alpha,\theta}(t/s)g(s;\alpha,\beta,\theta,\vartheta)\,ds.\end{aligned}$$

Let $0<\varepsilon<\min(\alpha,\beta,\pi\alpha/(2|\theta|))$. Using (3), we obtain

$$\begin{aligned}I(t)=&\frac{1}{2\pi i}\int_0^1\varphi_{\alpha,\theta}(-t/s)\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}f^{-}(z)s^{-z-1}\,dzds\\&+\frac{1}{2\pi i}\int_1^{\infty}\varphi_{\alpha,\theta}(-t/s)\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}f^{-}(z)s^{-z-1}\,dzds\\&+\frac{1}{2\pi i}\int_0^1\varphi_{\alpha,\theta}(t/s)\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}f^{+}(z)s^{-z-1}\,dzds\\&+\frac{1}{2\pi i}\int_1^{\infty}\varphi_{\alpha,\theta}(t/s)\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}f^{+}(z)s^{-z-1}\,dzds.\end{aligned}$$

In all integrals in the right hand side, we change the order of integration. This is possible by Fubini's theorem and (11). Hence, using (1), we have

(12)
$$\begin{aligned} I(t) =& \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} f^-(z)\int_0^1 \frac{s^{\alpha-z-1}}{s^\alpha+e^{-i\theta}t^\alpha}\,dsdz \\ &+\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} f^-(z)\int_1^\infty \frac{s^{\alpha-z-1}}{s^\alpha+e^{-i\theta}t^\alpha}\,dsdz \\ &+\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} f^+(z)\int_0^1 \frac{s^{\alpha-z-1}}{s^\alpha+e^{i\theta}t^\alpha}\,dsdz \\ &+\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} f^+(z)\int_1^\infty \frac{s^{\alpha-z-1}}{s^\alpha+e^{i\theta}t^\alpha}\,dsdz. \end{aligned}$$

Both of the integrals $\int_0^1 s^{\alpha-z-1}/(s^\alpha+e^{\pm i\theta}t^\alpha)\,ds$ converge uniformly on any compact set lying in $\{z : \operatorname{Re} z < \alpha\}$ and are bounded in $\{z : \operatorname{Re} z \le \varepsilon\}$. Hence, the integrations in the first and third integrals of (12) can be translated from $\{z : \operatorname{Re} z = -\varepsilon\}$ to $\{z : \operatorname{Re} z = \varepsilon\}$. Therefore (12) can be rewritten in the form:

(13)
$$\begin{aligned} I(t) =& \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} f^-(z)\int_0^\infty \frac{s^{\alpha-z-1}}{s^\alpha+e^{-i\theta}t^\alpha}\,dsdz \\ &+\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} f^+(z)\int_0^\infty \frac{s^{\alpha-z-1}}{s^\alpha+e^{i\theta}t^\alpha}\,dsdz. \end{aligned}$$

Using the equalities (4), (13) and

$$\int_0^\infty \frac{s^{\alpha-z-1}}{s^\alpha+e^{\pm i\theta}t^\alpha}\,ds = \frac{\pi}{\alpha}\frac{t^{-z}e^{\mp i\theta z/\alpha}}{\sin \pi z/\alpha}, \qquad 0 < \operatorname{Re} z < \alpha,$$

one can easily show that

(14)
$$I(t) = \frac{1}{2i\beta}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{t^{-z}e^{-i\vartheta z/\beta}}{\sin \pi z/\beta}\,dz.$$

The function

$$h(z) := \frac{1}{2i\beta}\frac{t^{-z}e^{-i\vartheta z/\beta}}{\sin \pi z/\beta}$$

is meromorphic with simple poles $\{q\beta\}_{q=-\infty}^{\infty}$. Evidently

(15)
$$\operatorname{Res}_{z=q\beta}(h(z)) = \frac{1}{2\pi i}(-1)^q t^{\beta q}e^{-i\vartheta q}, \qquad q \in \mathbb{Z}.$$

We will calculate the integral in (14) separately for $t > 1$ and $0 < t < 1$.

(a) $t > 1$. We apply the Cauchy residue theorem to the integral of $h(z)$ along the boundary of the region $\{z : \operatorname{Re} z > \varepsilon, |z| < (n + 1/2)\beta\}$ and then let $n \to \infty$. The integral along $C_n := \{z : \operatorname{Re} z \geq \varepsilon, |z| = (n + 1/2)\beta\}$ tends to 0 as $n \to \infty$ since

$$|\sin \pi z/\beta|^{-1} = O(e^{-\pi|\operatorname{Im} z|/\beta}) \text{ for } z \in C_n, n \to \infty,$$

and therefore

$$|h(z)| = \frac{1}{2\beta} e^{-\log t . \operatorname{Re} z} e^{|\vartheta \operatorname{Im} z|/\beta} |\sin \pi z/\beta|^{-1} = O(e^{-C|z|}) \quad \text{for } z \in C_n, n \to \infty,$$

where C is a positive constant. Using (15), we obtain

$$I(t) = -2\pi i \sum_{q=1}^{\infty} \operatorname{Res}_{z=q\beta}(h(z)) = \sum_{q=1}^{\infty} (-1)^{q+1} t^{\beta q} e^{-i\vartheta q} = \frac{1}{1 + e^{i\vartheta} t^{\beta}} = \varphi_{\beta,\vartheta}(t).$$

(b) $0 < t < 1$. Integrating the function $h(z)$ along the boundary of the region $\{z : \operatorname{Re} z < \varepsilon, |z| < (n + 1/2)\beta\}$ in a similar way as above, we obtain

$$I(t) = \sum_{q=0}^{\infty} (-1)^q t^{\beta q} e^{i\vartheta q} = \frac{1}{1 + e^{i\vartheta} t^{\beta}} = \varphi_{\beta,\vartheta}(t).$$

Thus, we have proved (2) for $t > 0$.

From (1), (3), (4) it is easy to derive the following equalities:

$$\varphi_{\beta,\vartheta}(t) = \varphi_{\beta,-\vartheta}(-t), \qquad g(s; \alpha, \beta, \theta, -\vartheta) = g(-s; \alpha, \beta, \theta, \vartheta).$$

Using them and the validity of (2) for $t > 0$, we obtain (2) for $t < 0$.

PROOF OF THEOREM 3. It suffices to prove that $g(s; \alpha, \beta, \theta, \vartheta)$ is non-negative. From (3), (4) we have

$$\begin{aligned} g(\pm s; \alpha, \beta, \theta, \vartheta) &= \frac{\alpha}{2\pi i \beta} \int_{-i\infty}^{i\infty} \frac{\sin \pi z/\alpha}{\sin \pi z/\beta} \frac{\sin z(\theta/\alpha \pm \vartheta/\beta)}{\sin 2z\theta/\alpha} s^{-z-1}\, dz, \\ &= \frac{\alpha}{2\pi s \beta} \int_{-\infty}^{\infty} \frac{\sinh \pi t/\alpha}{\sinh \pi t/\beta} \frac{\sinh t|\theta/\alpha \pm \vartheta/\beta|}{\sinh 2t|\theta|/\alpha} e^{-it \log s}\, dt, \end{aligned} \tag{16}$$

In the case when either $\alpha = \beta$, $|\vartheta|/\beta < |\theta|/\alpha$ or $\beta < \alpha$, $|\vartheta|/\beta = |\theta|/\alpha$, the assertion immediately follows from the fact (see, for example, [14, p. 35, 7.20]) that the function $\sinh by/\sinh b'y$ is a characteristic function up to a constant factor for $0 < b < b'$. In the case when simultaneously $\beta < \alpha$ and $|\vartheta|/\beta < |\theta|/\alpha$, we note that the function

$$\frac{\sinh \pi t/\alpha}{\sinh \pi t/\beta} \frac{\sinh t|\theta/\alpha \pm \vartheta/\beta|}{\sinh 2t|\theta|/\alpha} \tag{17}$$

is a characteristic function since it is a product of characteristic functions. Therefore the last integral in (16) is non-negative.

PROOF OF THE REMARK. It suffices to show that the function (17) is not a characteristic function under the conditions mentioned in the statement of the remark. It is easy to see that under these conditions the function (17) is analytic in the strip $\{t : |\operatorname{Im} t| < \min(\beta, \pi\alpha/2|\theta|)\}$ and has at least two imaginary zeros in it. This contradicts well-known properties of analytic characteristic functions (see, for example, [13, p. 29, Theorem 2.3.2 (a)]).

PROOF OF THEOREM 4. By [14, p. 35, 7.20],

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sinh \beta y}{\sinh \beta' y} e^{iyt}\, dy &= \frac{2\pi}{\beta'} \frac{e^{-\pi t/\beta'} \sin \beta\pi/\beta'}{1 + e^{-2\pi t/\beta'} + 2e^{-\pi t/\beta'} \cos \beta\pi/\beta'} \\ &= \frac{2\pi\beta}{\beta'} e^{-t} g(e^{-t}, \pi/\beta, \pi/\beta'), \qquad t \in \mathbb{R}.\end{aligned} \tag{18}$$

The second equality in (18) can easily be verified using the definition of $g(s, \alpha, \beta)$ given in (5). Proofs of (6), (7) are straightforward using (16) and (18).

To prove the last assertion of Theorem 4 note that if we substitute $s = e^{-\tau}$ in (16) we obtain

(19)
$$\frac{\beta}{\alpha} e^{-\tau} g(\pm e^{-\tau}; \alpha, \beta, \theta, \vartheta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh \pi t/\alpha}{\sinh \pi t/\beta} \frac{\sinh t|\theta/\alpha \pm \vartheta/\beta|}{\sinh 2t|\theta|/\alpha} e^{it\tau}\, dt, \quad \tau \in \mathbb{R}.$$

By using the convolution property of Fourier transforms, and (18) and (19), we have

$$\begin{aligned}&\frac{\beta}{\alpha} e^{-\tau} g(\pm e^{-\tau}; \alpha, \beta, \theta, \vartheta) \\ &= \frac{\theta\beta \pm \alpha\vartheta}{2\theta\alpha} \int_{-\infty}^{\infty} e^{-u} g(e^{-u}, \alpha, \beta) e^{-\tau+u} g(e^{-\tau+u}, \frac{\pi\alpha\beta}{\theta\beta \pm \vartheta\alpha}, \frac{\pi\alpha}{2\theta})\, du.\end{aligned} \tag{20}$$

Substituting $\tau = -\log s$ and $u = -\log v$ in (20) we obtain (8).

PROOF OF THEOREM 5. Evidently, $PD \backslash \{(1, \pi/2), (1, -\pi/2)\} = PD^{\star}_{1,\pi/2}$. Applying Theorem 2 with $\alpha = 1$, $\theta = \pi/2$ and noting that $\varphi_{1,\pi/2}(t) = 1/(1 + it)$ we obtain the representation (9) with $q(\pm s; \beta, \vartheta) = g(\pm s; 1, \beta, \pi/2, \vartheta)$. Using the equality (19), we obtain

$$\beta e^{-\tau} g(\pm e^{-\tau}; 1, \beta, \pi/2, \vartheta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh t|\pi/2 \pm \vartheta/\beta|}{\sinh \pi t/\beta} e^{it\tau}\, dt, \qquad \tau \in \mathbb{R}.$$

By [14, p. 35, 7.20], the function $(\sinh t|\pi/2 \pm \vartheta/\beta|)/(\sinh \pi t/\beta)$ is a characteristic function for all $(\beta, \vartheta) \in PD^{\star}_{1,\pi/2}$ and therefore $q(s; \beta, \vartheta)$ is a probability density and, moreover, the formula (10) holds.

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