A REMARK ON C-COMPACT SPACES

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Abstract

It has been observed by a number of researchers that although it is well-known that all continuous functions defined on C-compact spaces are closed functions, this property does not characterize C-compact spaces. In this note we employ the notion of strongly subclosed relations to prove that a space is C-compact if and only if all functions on it with strongly subclosed inverses are closed functions.

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Throughout this note all spaces are Hausdorff spaces. Let X be a space and let $A \subset X$. We denote the closure of A by $\overline{A}$ and the collection of open sets which contain A by $\Sigma(A)$ ($\Sigma(x)$ if $A = \{x\}$); we use the notation $\Gamma(x) = \{V - \{x\} : V \in \Sigma(x)\}$. The θ-*closure* of A, denoted by $cl_\theta A$, is $\bigcap_{\Sigma(A)} \overline{V}$ and the θ-*adherence* of a filterbase Ω, denoted by $ad_\theta \Omega$, is $\bigcap_\Omega cl_\theta F$. These notions were introduced by Veličko for the purpose of studying H-closed spaces and have subsequently received wide usage (see [1, 2]). A relation $F \subset X \times Y$ is *strongly subclosed* if $ad_\theta F(\Gamma(x)) \subset F(x)$ for each $x \in X$ for which $\Gamma(x)$ is a filterbase on X [1]. We will say that a function $g : X \to Y$ has a *strongly subclosed inverse* if the relation g^{-1} is strongly subclosed. It is not difficult to prove that continuous, and indeed θ-continuous [1], functions have strongly subclosed inverses.

A space X is said to be *C-compact* if for each closed $A \subset X$, each cover of A by open subsets of X contains a finite subfamily $\mathscr{V}$ such that $\{\overline{V} : V \in \mathscr{V}\}$ covers A. A space is H-*closed* if it is a closed subspace of every space in which it is embedded. It is known that a space X is C-compact if and only if each closed $A \subset X$ and filterbase Ω on A satisfy $A \cap ad_\theta \Omega \neq \emptyset$ and that a space X is H-closed if and only if every filterbase on X has nonempty θ-adherence [3]. We are now in a position to give a proof of the theorem.

THEOREM. *A space X is C-compact if and only if all functions on X with strongly subclosed inverses are closed functions.*

PROOF. Necessity. Let $A \subset X$ be closed, and $g : X \to Y$ have a strongly subclosed inverse. If y is a limit point of $g(A)$ then $\Omega = \{g^{-1}(W) \cap A : W \in \Gamma(y)\}$ is a filterbase on A and hence $\emptyset \neq A \cap ad_\theta(\Omega) \subset A \cap g^{-1}(y)$. So $g(A)$ is closed.

Sufficiency. Suppose A is a closed subset of X and that Ω is a filterbase on A such that $A \cap ad_\theta \Omega = \emptyset$. Since continuous functions have strongly subclosed inverses, it follows that X is H-closed and hence that $A \neq X$. Choose $v \in A$ and define $g : X \to Y$ by $g(x) = x$ if $x \in A$, $g(x) = v$ if $x \in X - A$, where $Y = X$ with the topology $\{V \subset X : v \in X - V$ or some $F \in \Omega$ satisfies $F \subset V\}$. Then Y is Hausdorff and $g^{-1} \subset g(X) \times X$ is strongly subclosed since $\Gamma(y)$ is a filterbase on Y only if $v = y$, and $ad_\theta g^{-1}(\Gamma(v)) \subset ad_\theta \Omega \subset X - A \subset g^{-1}(v)$. There is an $F_0 \in \Omega$ and $W \in \Sigma(v)$ with $\overline{W} \cap F_0 = \emptyset$ in X. It follows that $g(F_0) = F_0 \subset A - W = g(A - W)$, so $v \in \overline{A - W} - (A - W)$ in Y. Since $A - W$ is closed in X the function g is not a closed function. $\square$

References

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