

# EULER CHARACTERISTICS AND IMBEDDINGS OF HYPERBOLIC COXETER GROUPS

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## Abstract

This paper has a twofold purpose. The first is to compute the Euler characteristics of hyperbolic Coxeter groups  $W_S$  of level 1 or 2 by a mixture of theoretical and computer aided methods. For groups of level 1 and odd values of  $|S|$ , the Euler characteristic is related to the volume of the fundamental region of  $W_S$  in hyperbolic space. Secondly we note two methods of imbedding such groups in each other. This reduces the amount of computation needed to determine the Euler characteristics and also reduces the number of essentially different hyperbolic groups that need to be considered.

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## 0. Introduction

The Euler characteristic of a Coxeter group  $W_S$  was defined by Serre, who gave an inductive formula [13, p. 110] for its computation. More recently, Chiswell [4, Proposition 3] has derived the explicit formula

$$(0.1) \quad \chi(W_S) = \sum_{X \subset S} (-1)^{|X|} / |W_X|,$$

where the summation extends over all subsets  $X$  of  $S$  for which  $W_X$  is finite.

A crucial property of Euler characteristics is that

$$(0.2) \quad \chi(G) = [G : H]\chi(H),$$

whenever  $H$  is a subgroup of finite index in  $G$ . It follows in particular that  $\chi(W_S) = 1/|W_S|$  for finite  $W_S$ .

After preliminary discussion in Section 1, we show in Section 2 that  $\chi(W_S) = 0$  for affine  $W_S$  and then find, with the help of a *Mathematica* program, the Euler characteristics of ‘hyperbolic’ Coxeter groups in the sense of [7, 6.8], as well as of ‘hyperbolic groups of level 2’ in the sense of [11]. The numerical results are shown in Section 5.

It frequently happens that a Coxeter group  $W_{S'}$  can be naturally imbedded as a subgroup of finite index in another Coxeter group  $W_S$ . The value of  $\chi(W_{S'})$  then follows from (0.2). In Section 3, we describe two types of such imbeddings, some of which were observed earlier by the author in [11].

Finally we note in Section 4 that, for hyperbolic Coxeter groups, the volume of a fundamental region  $C$  of  $W_S$  in hyperbolic space  $\mathbb{H}^n$  is given by

$$(0.3) \quad (-1)^{n/2} \text{vol}(C) = \sigma_n \chi(W_S)/2,$$

whenever  $|S| = n + 1$  is odd ( $\sigma_n$  is the volume of the sphere  $S^n$ ). For even values of  $|S|$ ,  $\chi(W_S) = 0$  and gives no information about  $\text{vol}(C)$ .

Earlier computations of  $\text{vol}(C)$  were made by Meyerhoff [12] for  $|S| = 4$ . These can be considerably shortened by using the fact that

$$(0.4) \quad \text{vol}(C') = [W_S : W_{S'}] \text{vol}(C)$$

whenever  $W_{S'}$  is imbedded in  $W_S$ . We have checked that the calculations of [12] conform with equation (0.4) and list the volumes again in Table 1.

For  $|S| = 6$ , Kellerhals [10] has obtained an exact formula in one case, shown in Table 1. Using (0.4), we can deduce the volumes for related groups, two of which were also found in [10]. In addition, Kellerhals [9] has computed the volumes for ‘orthoschemes’, classified earlier by Im Hof [8], when  $|S|$  is odd. In view of (0.3), this amounts to finding their Euler characteristics. We have checked her values and found different answers in four cases. The second last graph on [9, p. 206] has a volume of  $61\pi^2/10800$ , the second and seventh graphs from the top of page 209 have volumes of  $\pi^3/259200$  and  $\pi^3/12960$  respectively, while the last graph on page 210 has a volume of  $17\pi^4/43545600$ .

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## 1. The Schläfli space

In this section, we establish certain combinatorial formulas whose origins lie in the work of Schläfli [1, 5.2] on volumes of simplices. When applicable, these formulas are helpful in reducing the amount of computation needed to find  $\chi(W_S)$ .

Let  $\Gamma$  be a finite graph and  $K$  a simplicial complex of subsets of  $\Gamma$ . We denote by  $K(X)$  the complex consisting of all proper subsets of a set  $X \subset \Gamma$ . If  $X_1, X_2$  are disjoint subsets of  $\Gamma$  such that no element of  $X_1$  is adjacent to an element of  $X_2$ , the union of  $X_1$  and  $X_2$  is written as  $X_1 \sqcup X_2$ . We also use this notation to indicate that  $X_1$  and  $X_2$  have this property.

The *Schläfli space*  $\mathbb{S}(K)$  of  $K$  is the real vector space spanned by elements  $[X]$ , for  $X \in K$ , subject to the relations:

(1) If  $A \sqcup X \in K$  and  $|X|$  is odd, then

$$(1.1) \quad 2[A \sqcup X] = \sum_{Y \in K(X)} (-1)^{|Y|} [A \sqcup Y].$$

We call  $E = \sum_{X \in K} (-1)^{|X|} [X]$  the *Euler element* of  $\mathbb{S}(K)$ .

A real-valued function  $\chi$  on  $K$  is a *Schläfli function* if it has the following properties:

- (2)  $\chi(\emptyset) = 1$ ;
- (3)  $\chi(X) = \chi(X_1)\chi(X_2)$  whenever  $X = X_1 \sqcup X_2$ ;
- (4)  $2\chi(X) = \sum_{Y \in K(X)} (-1)^{|Y|} \chi(Y)$  if  $|X|$  is odd.

It follows that  $\chi(\{x\}) = 1/2$  for all  $x \in \Gamma$ . For example, the function

$$(1.2) \quad \chi_1(X) = 1/2^{|X|}$$

always enjoys these properties.

A Schläfli function  $\chi$  has the same value on both sides of (1.1) and can therefore be extended to a linear function on  $\mathbb{S}(K)$ . We call

$$(1.3) \quad \chi(E) = \sum_{X \in K} (-1)^{|X|} \chi(X)$$

the *Euler characteristic* of  $\chi$ . Our aim is to find shorter formulas for  $E$ , in order to simplify the calculation of  $\chi(E)$ .

Suppose that  $A \sqcup X \in \Gamma$  is such that  $A \sqcup Y \in K$  for all  $Y \in K(X)$ . We can then construct the element

$$(1.4) \quad E(A, X) = \sum_{Y \in K(X)} (-1)^{|Y|} [A \sqcup Y]$$

of  $\mathbb{S}(K)$ . If  $A \sqcup X$  itself belongs to  $K$  and  $|X|$  is odd,  $E(A, X) = 2[A \sqcup X]$  by (1.1).

Let  $\{a_k\}$  be the sequence of numbers defined by  $a_1 = 1$  and

$$(1.5) \quad 2a_k = 2 - \binom{k}{1} a_1 - \cdots - \binom{k}{k-1} a_{k-1}.$$

LEMMA 1.1. *We have  $a_k = 4(2^{k+1} - 1)B_{k+1}/(k + 1)$ , where  $B_k$  is the  $k$ th Bernoulli number. In particular,  $a_k = 0$  for even values of  $k$ .*

PROOF. The recursion formula implies that

$$2 - \sum_{k=1}^{\infty} a_k t^k / k! = 4 / (e^t + 1).$$

On the other hand, starting from the equation

$$1 + \sum_{k=1}^{\infty} B_k t^k / k! = t / (e^t - 1),$$

substituting  $2t$  for  $t$  and subtracting, we obtain

$$2 - \sum_{k=1}^{\infty} (4(2^{k+1} - 1) B_{k+1} / (k+1)) t^k / k! = 4 / (e^t + 1).$$

since  $B_1 = -1/2$ .

PROPOSITION 1.2. *Suppose that  $A \sqcup X \subset \Gamma$  is such that  $A \sqcup Y \in K$  for all  $Y \in K(X)$ . Then*

$$E(A, X) = \sum_{Y \in K(X), |Y| \text{ even}} a_{|X \setminus Y|} [A \sqcup Y].$$

*In particular,  $E(A, X) = 0$  if  $|X|$  is even.*

PROOF. We argue by induction on  $|X|$ . If  $|Y|$  is odd, the term  $(-1)^{|Y|} [A \sqcup Y]$  on the right side of (1.4) is equal to  $-E(A, Y)/2$  by (1.1) and can therefore be expanded by the stated formula.

In the resulting expansion of  $E(A, X)$ , consider an element  $[A \sqcup Z]$ , with  $|Z|$  even. It occurs once on its own with a coefficient of  $+1$ . In addition, each element  $[A \sqcup Y]$ , with  $Z \subset Y \subset X$ ,  $Y \neq X$  and  $|Y|$  odd, contributes  $[A \sqcup Z]$  with a coefficient of  $-a_{2j+1}/2$ , where  $2j+1 = |Y \setminus Z|$ . Since there are  $\binom{|X \setminus Z|}{2j+1}$  such subsets of  $X$ , the total coefficient of  $[A \sqcup Z]$  is

$$1 - \sum_j \binom{|X \setminus Z|}{2j+1} a_{2j+1}/2 = a_{|X \setminus Z|}$$

by (1.5), which is equal to 0 if  $|X|$  is even.

COROLLARY 1.3. *Suppose that  $K = K(\Gamma)$ . Then  $E = 0$  if  $|\Gamma|$  is even and*

$$E = \sum_{k=0}^m a_{2k+1} \left( \sum_{|Y|=2m-2k} [Y] \right)$$

*if  $|\Gamma| = 2m + 1$  is odd.*

PROOF. Take  $A = \emptyset$  and  $X = \Gamma$  in Proposition 1.2.

For special graphs  $\Gamma$ , one can further simplify this formula by using

LEMMA 1.4. *Suppose that  $A \sqcup X \subset \Gamma$  is such that  $A \sqcup Y \in K$  for all  $Y \in K(X)$ . If  $K(X)$  can be expressed as the disjoint union of a family  $\{K_p\}$ , where  $K_p = \{B_p \sqcup Y' \mid Y' \subset C_p\}$  for a pair of subsets  $(B_p, C_p)$  of  $X$  such that  $B_p \sqcup C_p$ , we have*

$$E(A, X) = \sum_p (-1)^{|B_p|} [A \sqcup B_p \sqcup C_p].$$

PROOF. We have

$$\begin{aligned} \sum_{Y \in K_p} (-1)^{|Y|} [A \sqcup Y] &= (-1)^{|B_p|} \sum_{Y' \subset C_p} (-1)^{|Y'|} [A \sqcup B_p \sqcup Y'] \\ &= (-1)^{|B_p|} (E(A \sqcup B_p, C_p) + (-1)^{|C_p|} [A \sqcup B_p \sqcup C_p]), \end{aligned}$$

which is equal to  $(-1)^{|B_p|} [A \sqcup B_p \sqcup C_p]$  by Proposition 1.2. Summing over  $p$ , we obtain the desired result.

PROPOSITION 1.5. *Suppose that  $A \sqcup X \subset \Gamma$  is such that  $A \sqcup Y \in K$  for all  $Y \in K(X)$  and that  $X = \{1, \dots, n\}$  is a linear graph, with  $i$  adjacent to  $j$  only if  $i - j = \pm 1$ . Then*

$$E(A, X) = \sum_{j=1}^n (-1)^{j-1} \left[ A \sqcup \{1, \dots, \hat{j}, \dots, n\} \right].$$

PROOF. Apply Lemma 1.4 with  $B_i = \{1, \dots, i - 1\}$  and  $C_i = \{i + 1, \dots, n\}$  for  $1 \leq i \leq n$ .

COROLLARY 1.6. *Let  $\Gamma = \{1, \dots, 2m + 1\}$  be a linear graph, with  $i$  adjacent to  $j$  only if  $i - j = \pm 1$  and suppose that  $K = K(\Gamma)$ . Then*

$$E = \sum_{j=1}^{2m+1} (-1)^{j-1} \left[ 1, \dots, \hat{j}, \dots, 2m + 1 \right].$$

Other useful results of this type are

PROPOSITION 1.7. *Let  $\Gamma = \{1, \dots, 2m + 1\}$  be a cycle with  $i$  adjacent to  $j$  only if  $i - j = \pm 1 \pmod{2m + 1}$  and suppose that  $K = K(\Gamma)$ . Then*

$$E = \sum_{1 \leq i \leq j < 2m+1} (-1)^{j-i} \left[ 1, \dots, \hat{i}, \dots, \hat{j}, \dots, 2m + 1 \right].$$

PROOF. Apply Lemma 1.4 with  $B_{ij} = \{j + 1, \dots, 2m + 1, 1, \dots, i - 1\}$  and  $C_{ij} = \{i + 1, \dots, j - 1\}$ , for  $1 \leq i \leq j \leq 2m + 1$ .

PROPOSITION 1.8. *Let  $\Gamma$  be a linear graph  $\{1, \dots, n\}$ , with an extra vertex  $n + 1$  adjacent only to the vertex  $q$  and suppose that  $K = K(\Gamma)$ . Then*

$$E = \sum_{i=1}^{n+1} \epsilon(i) \left[ 1, \dots, \hat{i}, \dots, n + 1 \right] + \sum_{i=q+1}^n (-1)^{i-1} \left[ 1, \dots, \hat{i}, \dots, n \right],$$

where  $\epsilon(i) = (-1)^{i-1}$  for  $1 \leq i \leq q$  or  $i = n + 1$  and  $\epsilon(i) = (-1)^i$  for  $q + 1 \leq i \leq n$ .

PROOF. Let  $B_i = \{1, \dots, i - 1\}$ ,  $B'_i = \{1, \dots, i - 1, n + 1\}$ ,  $C_i = \{i + 1, \dots, n\}$  and  $C'_i = \{i + 1, \dots, n + 1\}$ . Now apply Lemma 1.4 with pairs  $(B_i, C'_i)$  for  $1 \leq i \leq q$ ,  $(B_i, C_i)$  for  $q + 1 \leq i \leq n + 1$ , and  $(B'_i, C_i)$  for  $q + 1 \leq i \leq n$ .

## 2. Euler characteristics of Coxeter groups

Suppose that  $W_S$  is a Coxeter group,  $\Gamma$  its Coxeter graph and  $K_W$  the simplicial complex consisting of all  $X \subset S$  such that  $W_X$  is finite.

Let

$$W_X(t) = \sum_{w \in W_X} t^{l(w)}$$

be the Poincaré series of a Coxeter group  $W_X$ . The formal identity [7, Prop. 5.12]

$$(2.1) \quad \sum_{Y \subset X} (-1)^{|Y|} / W_Y(t) = t^N / W_X(t) \quad (W_X \text{ finite})$$

$$(2.2) \quad = 0 \quad (W_X \text{ infinite}),$$

where  $N$  is the length of the element of largest length in  $W_X$ , enables one to calculate  $W_X(t)$  by induction on  $|X|$ . In particular, it follows that  $W_X(t)$  is a rational function of  $t$ , whose complex zeros are roots of unity other than  $t = 1$ ; therefore  $1/W_X(1)$  is a finite number.

When  $|X|$  is odd and  $W_X$  is finite, taking  $t = 1$  in (2.1) shows that

$$\sum_{Y \in K(X)} (-1)^{|Y|} / |W_Y| = 2 / |W_X|.$$

On the other hand, if  $W_X$  is finite and  $X = X_1 \sqcup X_2$ , then  $W_X$  is a direct product of  $W_{X_1}$  and  $W_{X_2}$ , so that

$$1 / |W_X| = 1 / |W_{X_1}| \cdot 1 / |W_{X_2}|.$$

These equations imply that

$$(2.3) \quad \chi(X) = 1/|W_X|$$

is a Schläfli function on  $K_W$  in the sense of Section 1. The Euler characteristic of  $\chi$  is equal, by (0.1), to the ‘Euler characteristic’  $\chi(W_S)$  of  $W_S$  in the sense of Serre [13]. We recall that

$$(2.4) \quad \chi(W_S) = 1/W_S(1)$$

by [13, Proposition 17].

**PROPOSITION 2.1.** *The Euler characteristic of an affine group  $W_S$  is equal to 0.*

**PROOF.** According to Bott’s theorem [2].

$$1/W_S(t) = \prod (1 - t^{m_i}) / W_Z(t),$$

where  $W_Z$  is a finite Coxeter group corresponding to  $W_S$  and  $\{m_i\}$  the set of exponents of  $W_Z$ . It follows from equation (2.4) that  $\chi(W_S) = 0$ .

Alternatively, one can note that  $W_S$  contains a normal subgroup of finite index isomorphic to  $\mathbb{Z}^n$ . Since  $\chi(\mathbb{Z}^n) = 0$ , we conclude from (0.2) that  $\chi(W_S)$  is also 0.

In [5], Coxeter gives a heuristic argument for this result and point out that it allows an inductive computation of the order of a finite Weyl group  $W_Z$  for both even and odd values of  $|Z|$ .

In [11], we have defined  $\Gamma$  to be of *level*  $\leq l$  if the deletion of any  $l$  vertices from  $\Gamma$  leaves the graph of a finite or an affine Coxeter group. If  $\Gamma$  is not also of level  $\leq l - 1$ , then  $l$  is called the level of  $W_S$ .

Groups of level 1 are the hyperbolic Coxeter groups in the sense of [7, 6.8]; they exist only for  $3 \leq |S| \leq 10$ . We extend  $\chi$  to a function  $\bar{\chi}$  on  $K(\Gamma)$  by letting  $\bar{\chi}(X) = 0$  whenever  $W_X$  is of affine type. It follows from Proposition 2.1 that  $\bar{\chi}$  remains a Schläfli function. Therefore  $\chi(W_S)$  can be computed as the Euler characteristic of  $\bar{\chi}$  by one of the formulas in Section 1; the results are listed in Table 1 below for  $|S| > 4$ . Since it is well known that

$$\chi(W_S) = 1/2m_{12} + 1/2m_{13} + 1/2m_{23} - 1/2 < 0$$

for  $|S| = 3$ , we conclude

**PROPOSITION 2.2.** *Suppose that  $W_S$  is a group of level 1. Then  $\chi(W_S) = 0$  for even values of  $|S|$ , while*

- (a)  $\chi(W_S) > 0$  if  $|S| = 1 \pmod{4}$ ;
- (b)  $\chi(W_S) < 0$  if  $|S| = 3 \pmod{4}$ .

Groups of level 2 exist only for  $4 \leq |S| \leq 11$  and are described in [11]. They are also ‘hyperbolic’ in the sense that the standard bilinear form associated to  $W_S$  is of signature  $(|S| - 1, 1)$ . In this case, we extend  $\chi$  to  $K(\Gamma)$  by defining, in addition,  $\bar{\chi}(X) = \chi(W_X)/2$  when  $W_X$  is hyperbolic. Again,  $\bar{\chi}$  remains a Schläfli function. If  $|S|$  is odd,  $\chi(W_S)$  can be computed as the Euler characteristic of  $\bar{\chi}$  by one of the formulas of Section 1. For even values of  $|S|$ , the latter is equal to 0 by Corollary 1.3 and therefore

$$(2.5) \quad \chi(W_S) = \sum_{X \in K(\Gamma), W_X \text{ hyperbolic}} \chi(W_X)/2.$$

In particular, it follows from Proposition 2.2 that  $\chi(W_S) < 0$  if  $|S| = 0 \pmod{4}$  and  $\chi(W_S) > 0$  if  $|S| = 2 \pmod{4}$ . The results are shown in Table 2 below for  $|S| \geq 5$ ; we observe

**PROPOSITION 2.3.** *Suppose that  $W_S$  is an irreducible group of level 2. Then*

- (a)  $\chi(W_S) > 0$  if  $|S| = 1, 2 \pmod{4}$ ;
- (b)  $\chi(W_S) < 0$  if  $|S| = 0, 3 \pmod{4}$ .

### 3. Imbeddings of Coxeter groups

Suppose that a Coxeter group  $W_{S'}$  is imbedded as a subgroup of finite index in another Coxeter group  $W_S$ . It follows from equation (0.2) that either

$$(3.1) \quad \chi(W_{S'}) = \chi(W_S) = 0,$$

or

$$(3.2) \quad \chi(W_{S'})/\chi(W_S) \in \mathbb{Z}$$

and is equal to the index of  $W_{S'}$  in  $W_S$ .

The following two results describe some standard ways of constructing imbeddings of Coxeter groups. Table 1 shows all the resulting imbeddings for groups of level 1. (Some of these were observed earlier by the author in [11].) In particular, we see that these groups fall into a relatively small number of commensurability classes.

**PROPOSITION 3.1.** *Suppose that the Coxeter graph  $\Gamma$  of  $W_S$  is a disjoint union  $\Gamma_1 \cup \Gamma_2$ , where*

- (i)  $\Gamma_1 = \{s_1, \dots, s_{l-1}\}$  is of type  $A_{l-1}$ ;
- (ii) there is exactly one edge  $(s_{l-1}, s_l)$  between the element  $s_{l-1} \in \Gamma_1$  and an element  $s_l \in \Gamma_2$  and no other edges between elements of  $\Gamma_1$  and  $\Gamma_2$ ;
- (iii)  $m(s_{l-1}, s_l)$  is an even number  $2m \geq 4$ .

Let  $S'$  be the set obtained from  $S$  by replacing  $s_k$  with

$$s'_k = s_k s_{k+1} \cdots s_{l-1} s_l s_{l-1} \cdots s_{k+1} s_k,$$

for some  $k$  such that  $1 \leq k \leq l-1$ .

Then  $W_{S'}$  is a Coxeter group with graph  $\Gamma' = \Gamma'_1 \cup \Gamma_2$ , where  $\Gamma'_1$  is obtained from  $\Gamma_1$  by replacing  $s_k$  with  $s'_k$  and joining  $s'_k$  to  $s_{k-1}$  (if  $k > 1$ ) with an edge marked by  $2m$ , and also to  $s_l$  with an edge marked by  $m$  (if  $m > 2$ ). Furthermore, a vertex  $s \in \Gamma_2$  joined to  $s_l$  with an edge marked by  $n$  is also joined to  $s'_k$  with an edge marked by  $n$ . The index of  $W_{S'}$  in  $W_S$  is equal to  $\binom{l}{k}$ .

PROOF. Using the notation of [3, V.4],  $s'_k$  is the reflection with normal

$$e'_k = e_l + 2 \cos(\pi/2m) \{e_{l-1} + \cdots + e_k\}.$$

Therefore  $B_M(e'_k, e_j)$  is equal to  $-\cos(\pi/m)$  for  $j = l$ , to  $-\cos(\pi/2m)$  for  $j = k-1$  and to 0 for other  $j$  in the range  $1 \leq j \leq l-1$ . On the other hand, for  $s \in \Gamma_2$ , we have  $B_M(e'_k, e_s) = B_M(e_l, e_s)$ .

If  $W'$  is the abstract Coxeter group with graph  $\Gamma'$ , these equations show that, with respect to the basis obtained from  $\{e_s\}$  by replacing  $e_k$  with  $e'_k$ , the form  $B_M$  provides the standard geometric realisation of  $W'$ . By a fundamental result of Tits [3, V.4.4, Corollary 2 of Theorem 1],  $W'$  is isomorphic to its image  $W_{S'}$ .

Consider a coset  $W_{S'}w$  of  $W_{S'}$  in  $W_S$ , with  $l(w)$  minimal. Using induction on  $l(w)$ , the relation  $s_k s_{k+1} \cdots s_l = s'_k s_k s_{k+1} \cdots s_{l-1}$  shows that  $w \in W_{\{s_1, \dots, s_{l-1}\}}$ . Since

$$W_{S'} \cap W_{\{s_1, \dots, s_{l-1}\}} = W_{\{s_1, \dots, \hat{s}_k, \dots, s_{l-1}\}},$$

we conclude that the cosets of  $W_{S'}$  in  $W_S$  correspond to the cosets of  $W_{\{s_1, \dots, \hat{s}_k, \dots, s_{l-1}\}}$  in  $W_{\{s_1, \dots, s_{l-1}\}}$  and that therefore  $[W_S : W_{S'}] = \binom{l}{k}$ .

In a similar way one can prove

**PROPOSITION 3.2.** *Suppose that the Coxeter graph  $\Gamma$  of  $W_S$  is a linear graph  $\{s_1, \dots, s_{l-1}, s_l, \dots, s_n\}$ , with all edges marked by 3, except for the edge between  $s_{l-1}$  and  $s_l$ , which is marked by  $3m \geq 6$ .*

*Let  $S'$  be the set obtained from  $S$  by replacing  $s_{l-1}$  with  $s'_{l-1} = s_l s_{l-1} s_l$  and  $s_l$  with  $s'_l = s_{l-1} s_l s_{l-1}$ . Then  $W_{S'}$  is a Coxeter group with graph  $\Gamma'$ , in which  $s'_{l-1}$  is joined*

to  $s_{l-2}$  (if  $l > 1$ ),  $s_{l+1}$  (if  $l < n$ ) and  $s'_l$  by edges marked with 3,  $3m$  and  $m$  (if  $m > 2$ ), respectively, while  $s'_l$  is joined to  $s_{l-2}$ ,  $s_{l+1}$  and  $s'_{l-1}$  with edges marked by  $3m$ , 3 and  $m$ .

The index of  $W_{S'}$  in  $W_S$  is equal to  $n + 1$ .

#### 4. Volumes of fundamental regions

Let  $\mathbb{S}^n$  be the unit sphere in  $\mathbb{R}^{n+1}$ . The volume  $\sigma_n$  of  $\mathbb{S}^n$  is equal to

$$(4.1) \quad \sigma_{2k} = \frac{2^{k+1} \pi^k}{1 \cdot 3 \cdots (2k-1)}, \quad \sigma_{2k+1} = \frac{2\pi^{k+1}}{k!},$$

for even and odd values of  $n$ , respectively.

Suppose that a finite Coxeter group  $W_S$ , with  $|S| = n + 1$ , acts on  $\mathbb{S}^n$ , rather than  $\mathbb{R}^{n+1}$ . The volume of its fundamental region  $C$  is given by

$$(4.2) \quad \text{vol}(C) = \sigma_n \chi(W_S),$$

since  $\mathbb{S}^n$  decomposes into  $|W_S|$  copies of  $\bar{C}$ .

On the other hand, a Coxeter group  $W_S$  of level 1, with  $|S| = n + 1$ , acts on the hyperbolic space  $\mathbb{H}^n$ . Copies of  $\bar{C}$  then cover the interior of  $\mathbb{H}^n$ , while elements  $s \in S$  for which  $W_{S \setminus \{s\}}$  is of affine type correspond to the vertices of  $\bar{C}$  which lie on the boundary of  $\mathbb{H}^n$ .

When  $|S|$  is odd, the volume of  $C$  is given by

$$(4.3) \quad (-1)^{n/2} \text{vol}(C) = \sigma_n \chi(W_S)/2.$$

(See [6] for a recent history of this formula.) This explains why the sign of  $\chi(W_S)$  must be as described in Proposition 2.2. However, if  $|S|$  is even, the Euler characteristic of  $W_S$  is zero and gives no information on the volume of  $C$ . The imbeddings of Coxeter groups shown in Table 1 at least reduce the number of cases to be considered, because of equation (0.4).

For groups of level 2, the fundamental region  $C$  is not contained in  $\mathbb{H}^n$  and there is no concept of ‘volume’, so that the significance of Proposition 2.3 remains unclear. However, such groups are related in [11] to packings of Euclidean space by unequal spheres.

#### 5. The tables

The subgroup relations for groups of level 1 obtained by methods of Section 3 are shown in Table 1. For each value of  $|S|$ , the groups are numbered in the same order as

TABLE 1. Groups of level 1

|            |  |
|------------|--|
| $ S  = 10$ | $3 \rightarrow_2 2, 1$   |
| $ S  = 9$  | $3 \rightarrow_2 2, 1, 4$<br>$\chi'(1) = 2, \chi'(2) = 17, \chi'(4) = 2^5 \cdot 17$  |
| $ S  = 8$  | $2 \rightarrow_2 1, 3, 4$  |
| $ S  = 7$  | $2 \rightarrow_2 1, 3$<br>$\chi'(1) = -7, \chi'(3) = -2^2 \cdot 13$  |
| $ S  = 6$  | $7 \rightarrow_2 5 \rightarrow_3 1, 9 \rightarrow_2 8 \rightarrow_3 4 \rightarrow_4 2 \rightarrow_5 1, 8 \rightarrow_2 6 \rightarrow_6 2,$<br>$4 \rightarrow_2 3 \rightarrow_{10} 1, 10, 11, 12$<br>$v(1) = 7\zeta(3)/46080 = 0.000183$  |
| $ S  = 5$  | $13 \rightarrow_2 11 \rightarrow_3 8 \rightarrow_2 6, 13 \rightarrow_2 12 \rightarrow_2 9 \rightarrow_3 6, 11 \rightarrow_2 9,$<br>$4 \rightarrow_2 1, 10 \rightarrow_2 7, 2, 3, 5, 14$<br>$\chi'(1) = 17, \chi'(2) = 2, \chi'(3) = 2^2 \cdot 13, \chi'(5) = 5 \cdot 11,$<br>$\chi'(6) = 5^2, \chi'(7) = 3 \cdot 5, \chi'(14) = 2^3 \cdot 5^2$   |
| $ S  = 4$  | $32 \rightarrow_2 16 \rightarrow_3 27 \rightarrow_4 26, 16 \rightarrow_2 28 \rightarrow_6 26, 12 \rightarrow_2 29 \rightarrow_5 26,$<br>$11 \rightarrow_5 27, 13 \rightarrow_2 26, 12 \rightarrow_2 14 \rightarrow_2 24, 29 \rightarrow_2 24,$<br>$7 \rightarrow_2 31 \rightarrow_2 23 \rightarrow_3 22, 5 \rightarrow_2 30 \rightarrow_2 22, 4 \rightarrow_2 1, 15 \rightarrow_2 25,$<br>$2, 3, 6, 8, 9, 10, 17, 18, 19, 20, 21$<br>$v(26) = L(\pi/3)/8 = 0.042289, v(22) = L(\pi/4)/6 = 0.076330,$<br>$v(1) = 0.035885, v(2) = 0.039050, v(3) = 0.093326,$<br>$v(6) = 0.556282, v(8) = 0.364107, v(9) = 0.525840,$<br>$v(10) = 0.672986, v(17) = 0.085770, v(18) = 0.222229,$<br>$v(19) = 0.358653, v(20) = 0.205289, v(21) = 0.502131,$<br>$v(25) = 0.171502$ |

in Section 6.9 of [7], by going down the first column and then (for  $|S| = 4, 6$ ) down the second. (The table in [7] actually contains an error: group number 12 for  $|S| = 5$  should have one of its edges marked by 4.) Notation such as  $m \rightarrow_k n$  means that group number  $m$  is a subgroup of group number  $n$  of index  $k$ . A group not involved in subgroup relations is mentioned as a single number  $n$ .

Since  $\chi(W_S)$  is, for these groups, a small rational number, it is convenient to scale it by a factor  $d_{|S|}$  given by

$$d_4 = 2^4 \cdot 3 \cdot 5, \quad d_5 = 2^7 \cdot 3^2 \cdot 5^2, \quad d_6 = 2d_5, \quad d_7 = 2^{10} \cdot 3^4 \cdot 5 \cdot 7,$$

$$d_8 = 2d_7, \quad d_9 = 2^{15} \cdot 3^5 \cdot 5^2 \cdot 7, \quad d_{10} = 2d_9, \quad d_{11} = 2^{18} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11.$$

When  $|S|$  is odd, the values of  $\chi'(W_S) = d_{|S|}\chi(W_S)$  are listed in Table 1 for the largest group in a commensurability class and can be deduced for the others by using

equation (0.2). For even values of  $|S|$ , we show instead the known volumes of the fundamental region  $C$ , taken from [10] and [12]. The letter  $L$  denotes the Lobachevsky function.

Table 2 gives the value of  $\chi'(W_S)$  for groups of level 2 when  $|S| > 4$ . The groups are taken in the same sequence as in [11, Table II]. Since the groups listed in brackets in that table can in fact be imbedded in the main group (and give rise to the same sphere packings), we do not list the value of  $\chi'(W_S)$  for them, but merely indicate their position by  $()$ .

TABLE 2. Groups of level 2

|            |  |
|------------|--|
| $ S  = 11$ | -33, -2, -2112   |
| $ S  = 10$ | 2, 4, 4, 1088, 68, 19, 1160  |
| $ S  = 9$  | 255, 1252, 164, 8704, 480, 4804, 1200, 12960   |
| $ S  = 8$  | -7, -14, -28, -14, -52, -104, -104, -312, -132   |
| $ S  = 7$  | -21, $()$ , -63, -196, -248, -684, -896, -2912, -882, -1015, -486, -2016, -1944, -5040   |
| $ S  = 6$  | 2, 17, 2, 4, 55, 55, 36, 120, 25, $()$ , 50, 50, 15, 15, 30, 60, 30, 200, 50, 150, 120, 90, 180, 65, 85, 34, 325, 400, 120, 170, 290   |
| $ S  = 5$  | 400, 85, 82, 182, 122, 375, 272, 507, 265, 165, 244, 240, 362, 555, 462, 724, 1200, 180, 735, 662, 842, 600, 842, 1920, 1560, 75, $()$ , 150, 200, 40, $()$ , 200, 125, 75, 240, 600, 122, 390, 480, 1800, 625, 300, 300, 325, 300, 240, 280, 375, 960, 482, 1440, 1200, 540, 625, 390, 675, 1250, 750, 964, 480, 750, 480, 900, 925, 975, 2400, 1082, 840, 962, 675, 1200, 1800, 865, 1200, 1920, 700, 2400, 2400, 780, 960, 1080, 1200, 900, 1200, 1500, 1080, 1275, 1200, 1585, 1825, 1440, 962, 1800, 2400, 2160, 2160 |

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