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PERIODIC SOLUTION OF THE CAUCHY PROBLEM

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Abstract

We derive necessary and sufficient conditions for the existence of a time-periodic solution to the abstract Cauchy problem.

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1. Introduction

We study the existence of time-periodic solution to the differential equation

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + f(t), \quad t \geq 0, \\ u(0) &= x, \end{aligned} \tag{1.1}$$

where A is the infinitesimal generator of an eventually norm continuous semigroup $T(t)$ and f is a continuous function in a Banach space X. We say f is *w-periodic* if w is the infimum of the set of all $\tau > 0$ such that $f(t) = f(t+\tau)$ for all $t \geq 0$. If f is w-periodic then, by uniqueness, $u(\cdot)$ the mild solution of (1.1) is w-periodic if and only if $u(0) = u(w)$. We say (1.1) has a *w-periodic solution* if there exists $x \in X$ such that

$$x = u(w) = T(w)x + \int_0^w T(w-s)f(s)\,ds.$$

When A is the infinitesimal generator of a C_0-semigroup $T(t)$, it was shown in Prüss [4] that (1.1) admits a unique w-periodic solution for any given w-periodic continuous function f if and only if 1 is not in the spectrum of $T(w)$.

The aim of this paper is to derive necessary and sufficient conditions for existence of periodicity when A is the infinitesimal generator of an eventually norm continuous semigroup $T(t)$ and 1 is a pole of $T(w)$. Our conditions do not rely on explicit knowledge of the semigroup, but only of its generator.

In a Banach space setting, when $T(t)$ is a C_0-semigroup generated by A and 1 is a pole of order greater than one, the problem of existence of a periodic solution to (1.1) is non-trivial, since 1 being a simple pole of $T(w)$ (pole of order one) characterizes a w-periodic semigroup. This observation can be deduced from Engel [2, Theorem IV 2.26, Corollary IV 3.8], and the decomposition theorem that characterizes poles. In Straškraba [5], the general case of isolated spectral points was considered for a self-adjoint generator of a C_0-semigroup in a Hilbert space. More results on periodic solutions to abstract evolution problems were obtained in Daners [1].

Let A be a closed linear operator in a Banach space X. The set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is invertible is called the *resolvent set* of A, denoted by $\rho(A)$. The complement of $\rho(A)$ in $\mathbb{C}$ is the *spectrum* $\sigma(A)$ of A. We call μ a *pole of* A if μ is a pole of $(\lambda I - A)^{-1}$. A bounded linear operator A is *nilpotent of order* $k \in \mathbb{N}$ if $A^k = 0$ and $A^n \neq 0$ for all $n < k$. For relevant facts from the operator theory of linear operators, see Kato [3] and Taylor [6].

A C_0-semigroup $T(t)$ is *eventually norm continuous* if there exists $t_0 \geq 0$ such that $T(t)$ is norm continuous for all $t > t_0$. We call a C_0-semigroup $T(t)$ w-*periodic* if there exists $t_0 > 0$ such that $T(t_0) = I$ and

$$w = \inf_{t_0 > 0} \{T(t_0) = I\}.$$

For relevant facts and properties of eventually norm continuous semigroups and periodic C_0-semigroups, see Engel [2].

2. Necessary and sufficient conditions for periodic solutions

Let $T(t)$ be an eventually norm continuous semigroup generated by A and 1 be a pole of $T(w)$. We give necessary and sufficient conditions that ensure (1.1) has a periodic solution. The conditions only depend on the knowledge of the generator. We first need two propositions. For $j \in \mathbb{N}$, the function $F^{(j)}$ is a j-th primitive of f if $dF^{(n)}(t)/dt = F^{(n-1)}(t)$ for each natural number $n \leq j$ and $F^{(0)}(t) = f(t)$.

PROPOSITION 2.1. *Let* A *be a nilpotent operator of order* $k+1$, *where* $k \in \mathbb{N}$. *If* $\int_0^w f(t)dt = 0$ *then* (1.1) *has a* w-*periodic solution.*

PROOF. Firstly we observe that for each $j \in \mathbb{N}$ there exists a j-th primitive of f

such that $F^{(j)}(w) = F^{(j)}(0)$. For if $\int_0^w F^{(j-1)}(t)dt = d \neq 0$, let

$$H^{(j-1)}(t) = F^{(j-1)}(t) - \frac{d}{w}.$$

Then $\int_0^w H^{(j-1)}(t)dt = 0$, and $dH^{(j-1)}(t)/dt = dF^{(j-1)}(t)/dt$. Hence $H^{(j-1)}$ is a $(j-1)$-th primitive of f and $H^{(j)}(w) = H^{(j)}(0)$. Secondly we observe that $u(t) = \sum_{j=1}^{k} A^j F^{(j)}(t)$ satisfies System (1.1) if $x = \sum_{j=1}^{k} A^j F^{(j)}(0)$. Therefore $u(0) = u(w)$. □

PROPOSITION 2.2. *If A is a nilpotent operator of order $k+1$ then (1.1) has a w-periodic solution if and only if*

$$Ax = \sum_{n=1}^{k-1} A^{n+1} G^{(n)}(0) - \frac{1}{w}\int_0^w f(t)dt,$$

where $G^{(n)}$ is the n-th primitive of g such that $G^{(n)}(w) = G^{(n)}(0)$, and

$$g(t) = f(t) - \frac{1}{w}\int_0^w f(t)dt.$$

PROOF. Let $\int_0^w f(t)dt = c$ and $g(t) = f(t) - c/w$. Then $\int_0^w g(t)dt = 0$. By Proposition 2.1, the equation

$$\begin{aligned} \frac{dv(t)}{dt} &= Av(t) + g(t), \quad t \geq 0, \\ v(0) &= v_0, \end{aligned} \tag{2.1}$$

has a w-periodic solution if $v_0 = \sum_{j=1}^{k} A^j G^{(j)}(0)$.

Now let $u(t)$ be the solution of (1.1) with $f(t) = g(t) + c/w$ and $v(t)$ be the solution of (2.1). Put $y(t) = u(t) - v(t)$. Then $y(t)$ is the solution of

$$\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) + \frac{c}{w}, \quad t \geq 0, \\ y(0) &= y_0 = x - v_0. \end{aligned} \tag{2.2}$$

Since A is nilpotent of order $k+1$, we have

$$y(t) = \exp(At)y_0 + \int_0^t \sum_{n=0}^{k} \frac{A^n}{n!}(t-s)^n \frac{c}{w} ds,$$

We can therefore express $y(t)$ as a polynomial in t

$$y(t) = y_0 + \left(Ay_0 + \frac{c}{w}\right)t + \left(\frac{A^2 y_0}{2!} + \frac{A(w^{-1}c)}{2!}\right)t^2 + \cdots.$$

Since (2.1) has a periodic solution (when $v_0 = \sum_{j=1}^{k} A^j G^{(j)}(0)$), Equation (2.2) has a periodic solution if and only if $Ay_0 + c/w = 0$. This completes the proof. □

We can now prove our main theorem.

THEOREM 2.3. *Let A be the infinitesimal generator of an eventually norm continuous semigroup $T(t)$ and 1 be a pole of order $k+1$ of $T(w)$ with the spectral projection P. Then there exists a bounded subset J of $\mathbb{Z}$ such that $P = \sum_{j\in J} P_j$ and $P_jP_k = \delta_{jk}P_j$, where P_j is the spectral projection of A at the pole $(2\pi i/w)j$, and δ_{jk} is the Kronecker symbol. Let A_j be the restriction of A to P_jX and $B_j = A_j - (2\pi i/w)jI$. Then (1.1) has a w-periodic solution if and only if for each $j \in J$*

$$B_jP_jx = \sum_{n=1}^{k-1} B_j^{n+1}G_j^{(n)}(0) - \frac{1}{w}\int_0^w \exp\left(-\frac{2\pi i}{w}jt\right) P_jf(t)\,dt,$$

where $G_j^{(n)}$ is the n-th primitive of

$$P_jg(t) = \exp\left(-\frac{2\pi i}{w}jt\right) P_jf(t) - \frac{1}{w}\int_0^w \exp\left(-\frac{2\pi i}{w}jt\right) P_jf(t)\,dt,$$

such that $G_j^{(n)}(w) = G_j^{(n)}(0)$.

PROOF. On the subspace $(I-P)X$, (1.1) has a unique w-periodic solution since 1 is in the resolvent set of the restriction of $T(w)$ to $(I-P)X$. The existence of a finite subset J of $\mathbb{Z}$ such that $P = \sum_{j\in J} P_j$ and $P_jP_k = \delta_{jk}P_j$ is a direct consequence of Engel [2, Theorem II.4.18]. Further, it follows from Engel [2, Page 283] that on each PX_j, the point $(2\pi i/w)j$ is a pole of maximal order $k+1$.

On each P_jX observe that,

$$\begin{aligned} \frac{du_j(t)}{dt} &= A_ju_j(t) + f_j(t), \quad t \geq 0, \\ u_j(0) &= P_jx, \end{aligned} \tag{2.3}$$

has a w-periodic solution if and only if

$$\begin{aligned} \frac{du_j(t)}{dt} &= B_ju_j(t) + \exp\left(-\frac{2\pi i}{w}jt\right) f_j(t), \quad t \geq 0, \\ u_j(0) &= P_jx, \end{aligned} \tag{2.4}$$

has a w-periodic solution. This can be seen through the identities

$$\begin{aligned} P_jx &= \exp\left(A_jw\right) P_jx + \int_0^w \exp\left(A_j(w-s)\right) f_j(s)\,ds \\ &= \exp\left(B_jw\right) P_jx + \int_0^w \exp\left(B_j(w-s)\right) \exp\left(-\frac{2\pi i}{w}js\right) f_j(s)\,ds. \end{aligned}$$

We can complete the proof by applying Proposition 2.2 to Equation (2.4). □

When 1 is a simple pole of $T(w)$, we have the following result.

COROLLARY 2.4. *Let A be the infinitesimal generator of an eventually norm continuous semigroup $T(t)$ and 1 be a simple pole of $T(w)$. Then (1.1) has a w-periodic solution if and only if*

$$\int_0^w \sum_{j\in J} \exp\left(-\frac{2\pi i}{w} js\right) P_j f(s)\, ds = 0,$$

where J is a finite subset of $\mathbb{Z}$ and P_j is the spectral projection of A at $(2\pi i/w)j$.

When the range of f is restricted in $\mathscr{D}(A)$, the domain of A, we have a similar result to Corollary 2.4 for general C_0-semigroups.

THEOREM 2.5. *Let $T(t)$ be a C_0-semigroup generated by A and 1 be a simple pole of $T(w)$. If $f(t) \in \mathscr{D}(A)$ for all $t \geq 0$ then (1.1) has a w-periodic solution in $\mathscr{D}(A)$ if and only if*

$$\int_0^w \sum_{n=-\infty}^{\infty} \exp\left(-\frac{2\pi i}{w} ns\right) P_n f(s) ds = 0,$$

where P_n is the spectral projection of A at $(2\pi i/w)n$.

PROOF. Let P be the spectral projection of $T(w)$ at 1. We can write $T(t) = T_1(t) \oplus T_2(t)$, where $T_1(t)$ and $T_2(t)$ are C_0-semigroups generated by A_1 and A_2, the restrictions of A to the invariant subspaces PX and $(I-P)X$, respectively. On $(I-P)X$, 1 is in $\rho(T_2(w))$, thus (1.1) has a unique w-periodic solution. On PX, since 1 is a simple pole of $T_1(w)$, the spectrum of A_1 consists of at most simple poles at $(2\pi i/w)n$, $n \in \mathbb{Z}$ (see Engel [2, Page 283]). By Engel [2, Theorem IV.2.26], $T_1(t)$ is a w-periodic C_0-semigroup, that is $T_1(0) = T_1(w)$, and for $f(t) \in \mathscr{D}(A)$, $t \geq 0$, the mild solution of (1.1) on PX is

$$\begin{aligned} u_1(t) &= T_1(t)Px + \int_0^t T_1(t-s)Pf(s)\, ds \\ &= T_1(t)Px + \int_0^t \sum_{n=-\infty}^{\infty} \exp\left(\frac{2\pi i}{w} n(t-s)\right) P_n f(s)\, ds. \end{aligned}$$

Since $T_1(0) = T_1(w)$, $u_1(0) = u_1(w)$ if and only if

$$\int_0^w \sum_{n=-\infty}^{\infty} \exp\left(-\frac{2\pi i}{w} ns\right) P_n f(s)\, ds = 0. \qquad \square$$

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