PERMUTABLE FUNCTIONS CONCERNING DIFFERENTIAL EQUATIONS

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Abstract

Let \( f \) and \( g \) be two permutable transcendental entire functions. Assume that \( f \) is a solution of a linear differential equation with polynomial coefficients. We prove that, under some restrictions on the coefficients and the growth of \( f \) and \( g \), there exist two non-constant rational functions \( R_1 \) and \( R_2 \) such that \( R_1 \cdot f = R_2 \cdot g \). As a corollary, we show that \( f \) and \( g \) have the same Julia set: \( J(f) = J(g) \). As an application, we study a function \( f \) which is a combination of exponential functions with polynomial coefficients. This research addresses an open question due to Baker.

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1. Introduction and Main Results

Let \( f \) be a meromorphic function. We denote by \( T(r, f) \) the Nevanlinna characteristic of \( f \). The order and the lower order of \( f \) are defined by

\[
\lambda(f) = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}
\]

and

\[
\rho(f) = \liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r} = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r},
\]

respectively, where \( M(r, f) = \max\{|f(z)| : |z| = r\} \) is the maximum modulus (see for example [8] for an introduction to Nevanlinna Theory).

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Let $f$ and $g$ denote two meromorphic functions. If

\[(1.1) \quad f(g) = g(f),\]

then we call $f$ and $g$ permutable. Many mathematicians have studied the analytic and dynamical properties of $f$ and $g$. The following general results are known.

- For a given $f$, there exist infinitely many transcendental entire functions $g$ such that $f(g) = g(f)$, for example, $g = f^n$ will do, where $f^n$ denotes the $n$-th iterate of $f$: $f^n = f^{n-1}(f)$. There should be no confusion with ordinary powers, which will be explicitly written as $f^n(z)$ if necessary.

- For a given $f$, there are only countably many entire functions $g$ such that (1.1) holds (see [2]).

- Let $f(z) = a e^{bz} + c \ (ab \neq 0, a, b, c \in \mathbb{C})$. If $f(g) = g(f)$ then $g = f^n$ for some $n \geq 0$ (see [1]).

In this paper we shall study relations between permutable entire functions and differential equations. In fact, we shall consider functions which are solutions of some linear differential equations of the form

\[(1.2) \quad p_n(z)f^{(n)}(z) + p_{n-1}(z)f^{(n-1)}(z) + \cdots + p_1(z)f'(z) + p_0(z)f(z) + p(z) = 0,
\]

where $n$ is a positive integer, and $p_i (0 \leq i \leq n)$ and $p$ are polynomials, with $p_n \neq 0$.

**THEOREM 1.1.** Let $f$ and $g$ be two permutable transcendental entire functions with $\rho(f) > 0$ and $\lambda(g) < \infty$. If

(i) $f(z)$ satisfies (1.2) with $p_0(z) \neq 0$ and $p(z)/p_0(z) \neq$ constant;

(ii) $f(z)$ cannot be a solution of any linearly differential equation of order $\leq n - 1$ with polynomial coefficients,

then there exist two nonconstant rational functions $R_1(z)$ and $R_2(z)$ such that $R_1(f) \equiv R_2(g)$.

As an application, we consider the following function $f(z)$:

\[(1.3) \quad f(z) = p(z) + p_1(z)e^{p_1(z)} + p_2(z)e^{p_2(z)} + \cdots + p_n(z)e^{p_n(z)},\]

where $p(z)$ is a polynomial, $p_i(z) \ (i = 1, \ldots, n)$ are non-zero polynomials and $q_i(z) \ (i = 1, \ldots, n)$ are polynomials with $q_i(z) - q_j(z) \neq$ constant for $1 \leq i \neq j \leq n$.

**THEOREM 1.2.** Let $f$ and $g$ be two permutable transcendental entire functions with $\lambda(g) < \infty$, where $f$ satisfies (1.3). Assume that $p(z)$ is not a constant. Then there exist two rational functions $P_1(z)$ and $P_2(z)$ such that $P_1(f) = P_2(g)$. 

REMARK. In [17], we studied a special case where \( n \leq 2 \).

Next, we list some well-known permutable transcendental entire functions of exponential type (see Ng [14] for other examples).

**Example 1.** Let \( f(z) = z + \gamma e^z \) and \( g(z) = z + \gamma e^z + 2k\pi i \), where \( \gamma \neq 0 \in \mathbb{C} \) and \( k \in \mathbb{Z} \). Then \( f \circ g = g \circ f \).

**Example 2.** Let \( g_1(z) = z + \gamma \sin z + 2k\pi \) and \( g_2(z) = -z - \gamma \sin z + 2k\pi \) and \( f(z) = z + \gamma \sin z \), where \( \gamma \neq 0 \in \mathbb{C} \) and \( k \in \mathbb{Z} \). Then \( f \circ g_1 = g_1 \circ f \) and \( f \circ g_2 = g_2 \circ f \).

**Example 3.** Let \( f(z) = ia \left[ \exp \left( \frac{(4k+3)\pi}{8a^2} iz^2 \right) + \exp \left( -\frac{(4k+3)\pi}{8a^2} iz^2 \right) \right] \),

\[
g(z) = a \left[ \exp \left( \frac{(4k+3)\pi}{8a^2} iz^2 \right) - \exp \left( -\frac{(4k+3)\pi}{8a^2} iz^2 \right) \right], \quad q(z) = \frac{(4k+3)\pi}{8a^2} iz^2,
\]

where \( a \in \mathbb{C} \), \( a \neq 0 \) and \( k \in \mathbb{N} \). It is easy to check that \( q(g) = -q(f) - (2k + 1.5)\pi i \) and \( f(g) = g(f) \).

The motivation for this research comes from the following open question in complex dynamics.

Let \( f \) be a nonlinear meromorphic function. We define

\[
F = F(f) = \{ z \in \overline{\mathbb{C}} : \text{the sequence } \{f^n\} \text{ is well defined and normal at } z \}
\]

and

\[
J = J(f) = \overline{\mathbb{C}} - F(f),
\]

where \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \), and the concept normal is in the sense of Montel. \( F(f) \) and \( J(f) \) are called the Fatou and Julia sets of \( f \), respectively. When there is no confusion, we briefly write \( F \) and \( J \) instead of \( F(f) \) and \( J(f) \). Clearly \( F(f) \) is open and it is well-known that \( J(f) \) is a nonempty perfect set which either coincides with \( \mathbb{C} \) or is nowhere dense in \( \mathbb{C} \). For the basic results in the dynamical system theory of transcendental functions, we refer the reader to the books [9] and [13].

**Open Question 1 (Baker [1]).** For two given permutable transcendental entire functions \( f \) and \( g \), does it follow that \( F(f) = F(g) \)?

This is a difficult question to answer. So far, affirmative answers to special cases of functions of \( f \) and \( g \) have been obtained (see [1, 16, 18]). When \( f \) and \( g \) are permutable rational functions, Fatou [4, 5, 6] and Julia [10] proved that they have the same Julia set.
**Corollary 1.3.** Let $f$ and $g$ satisfy the assumptions of Theorem 1.1 or Theorem 1.2. Then $J(f) = J(g)$.

2. Some Lemmas

**Lemma 2.1 ([7]).** Let $G_0, G_1, \ldots, G_m$ and $f$ be non-constant entire functions and let $h_0, h_1, \ldots, h_m (m \geq 1)$ be nonzero meromorphic functions. Suppose that $K$ is a positive number and $\{r_j\}$ is an unbounded monotone increasing sequence of positive numbers such that, for each $j \geq 1$, 

$$T(r_j, h_i) \leq KT(r_j, f) \quad (i = 0, \ldots, m)$$

and 

$$T(r_j, f') \leq (1 + o(1))T(r_j, f).$$

If 

$$h_0G_0(f) + h_1G_1(f) + \cdots + h_mG_m(f) \equiv 0$$

then there exist polynomials $\{p_j\} (j = 0, 1, \ldots, m)$, not all identically zero, such that 

$$p_0(z)G_0(z) + p_1(z)G_1(z) + \cdots + p_m(z)G_m(z) \equiv 0.$$

**Lemma 2.2 ([3]).** Let $f_j(z) (j = 1, 2, \ldots, n)$ and $g_j(z) (j = 1, 2, \ldots, n, n \geq 2)$ be two systems of entire functions satisfying the following conditions:

1. $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
2. for $1 \leq j, k \leq n, j \neq k$, $g_j(z) - g_k(z)$ is non-constant;
3. for $1 \leq h, k \leq n, h \neq k, 1 \leq j \leq n, T(r, f_j) = o[T(r, e^{g_h-r_k})]$.

Then $f_j(z) \equiv 0 \ (j = 1, 2, \ldots, n)$.

To state the following result, we denote by $W(f_1, f_2, \ldots, f_n)$ the Wronskian of the functions $f_1, \ldots, f_n$:

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

**Lemma 2.3 ([15], Problem 60, Chapter VII).** Let $f_j(z) (j = 1, 2, \ldots, n)$ be transcendental entire functions. If $W(f_1, \ldots, f_n) \equiv 0$ but $W(f_1, \ldots, f_{n-1}) \neq 0$, then there exist constants $c_1, c_2, \ldots, c_{n-1}$ such that 

$$f_n(z) = c_1f_1(z) + c_2f_2(z) + \cdots + c_{n-1}f_{n-1}(z).$$
This implies the following lemma.

**Lemma 2.4.** If \( f_j(z) (j = 1, 2, \ldots, n) \) are linearly independent transcendental entire functions then \( W(f_1, \ldots, f_n) \not\equiv 0 \).

**Lemma 2.5 ([19]).** Let \( f \) and \( g \) be two permutable entire functions satisfying

\[
0 < \rho(f) < \lambda(f) < \infty, \quad \lambda(g) < \infty.
\]

Then for any given positive integer \( n \) there exist a positive constant \( K \) and a sequence \( \{r_j\} \) tending to \( \infty \) such that

\[
T(r_j, g^{(i)}) \leq KT(r_j, f)
\]

for all \( j \geq 1 \) and \( 0 \leq i \leq n \).

**Lemma 2.6 ([12]).** If \( f \) and \( g \) are two permutable transcendental entire functions and there exists a nonconstant polynomial \( \Phi \) in both \( x \) and \( y \) such that

\[
\Phi(f, g) \equiv 0,
\]

then \( J(f) = J(g) \).

**Lemma 2.7.** Let \( n \geq 1 \), \( f \) and \( g \) be two permutable transcendental entire functions with \( \rho(f) > 0 \) and \( \lambda(g) < \infty \), and let \( p_i(z) (0 \leq i \leq n) \) and \( p(z) \) be polynomials. If

(1) \( p_n(z) \not\equiv 0 \) and \( p_0(z) \not\equiv 0 \),

(2) \( f(z) \) satisfies the following differential equation:

\[
(2.1) \quad p_n(z)f^{(n)}(z) + p_{n-1}(z)f^{(n-1)}(z) + \cdots + p_1(z)f'(z) + p_0(z)f(z) + p(z) = 0,
\]

(3) \( f(z) \) cannot be a solution of any linearly differential equation with polynomial coefficients of order \( \leq n - 1 \),

then there exist four polynomials \( Q(z), Q_0(z), Q_{n-1}(z) \) and \( Q_n(z) \), with \( Q_0(z) \not\equiv 0 \) and \( Q_n(z) \not\equiv 0 \) such that

\[
Q_{n-1}(f)p_n(g) \left( \frac{f'}{g'} \right)^n - Q_n(f) p_n(g) \left[ p_n(g) \frac{a_n f^{n-2}(f^{n} g' - f'g'')}{(g')^{n+1}} + p_{n-1}(g) \left( \frac{f'}{g'} \right)^{n-1} \right] = 0,
\]

(2.2) \( Q_0(f) p_n(g) \left( \frac{f'}{g'} \right)^n - Q_n(f) p_0(g) = 0 \)

and

(2.3) \( Q(f) p_n(g) \left( \frac{f'}{g'} \right)^n - Q_n(f) p(g) = 0 \),

where \( a_1 = a_2 = 1 \) and \( a_n = n(n-1)/2 \) for \( n \geq 3 \).
PROOF. From (2.1) we see that \( \lambda(f) < \infty \) (see [11]). By (1.1) we have

\[
f''(g) g' = g'(f) f',
\]

\[
f''(g) g^2 + f'(g) g'' = g'(f) f'^2 + g'(f) f'',
\]

\[
f'''(g) g^3 + 3 f''(g) g^2 g' + f'(g) g''' = g'''(f) f'^3 + 3 g''(f) f' f'' + g'(f) f''',
\]

\[
f^{(4)}(g) g^4 + 6 f'''(g) g^2 g' + f''(g) A_{4,2}(g', g', g'') + f'(g) g^{(5)}
\]

\[
= g^{(4)}(f) f'^{4} + 6 g'''(f) f'^2 f'' + g''(f) B_{4,2}(f', f'', f''') + g'(f) f^{(5)},
\]

\[
f^{(5)}(g) g^5 + 10 f^{(4)}(g) g^3 g'' + f'''(g) A_{5,3}(g', \ldots, g^{(4)}) + f''(g) A_{5,2}(g', \ldots, g^{(4)})
\]

\[
+ f'(g) g^{(6)}
\]

\[
= g^{(5)}(f) f'^{5} + 10 g^{(4)}(f) f'^3 f'' + g''(f) B_{5,3}(f', \ldots, f^{(4)})
\]

\[
+ g''(f) B_{5,2}(f', \ldots, f^{(4)}) + g'(f) f^{(5)},
\]

\[
\ldots
\]

\[
f^{(n)}(g)(g')^n + f^{(n-1)}(g)a_n(g')^{n-2} g'' + f^{(n-2)}(g) A_{n,n-2}(g', \ldots, g^{(n-1)}) + \ldots
\]

\[
+ f'(g) A_{n,2}(g', \ldots, g^{(n-1)}) + f'(g) g^{(n)}
\]

\[
= g^{(n)}(f) (f')^n + g^{(n-1)}(f)a_n(f')^{n-2} f'' + g^{(n-2)}(f) B_{n,n-2}(f', \ldots, f^{(n-1)}) + \ldots
\]

\[
+ g''(f) B_{n,2}(f', \ldots, f^{(n-1)}) + g'(f) f^{(n)}
\]

where \( A_{i,j}(g', \ldots, g^{(i-2)}) (i \geq 3, 2 \leq j \leq i - 2) \) are polynomials of \( g', g'', \ldots, g^{(i-2)} \) and \( B_{i,j}(f', \ldots, f^{(i-2)}) (i \geq 3, 2 \leq j \leq i - 2) \) are polynomials of \( f', f'', \ldots, f^{(i-2)} \).

Solving the above system yields

\[
f'(g) = \frac{f'}{g'} g'(f),
\]

\[
f''(g) = \left( \frac{f'}{g'} \right)^2 g''(f) + \left[ \frac{f''}{g'^2} - \frac{f' g''}{g^3} \right] g'(f),
\]

\[
f'''(g) = \left( \frac{f'}{g'} \right)^3 g'''(f) + \left[ \frac{3 f' (f' g' - f'' g'')}{g^4} \right] g''(f) + C_{3,1}(f', g') g'(f),
\]

\[
f^{(4)}(g) = \left( \frac{f'}{g'} \right)^4 g^{(4)}(f) + \left[ \frac{6 (f')^2 (f' g' - f'' g'')}{g^5} \right] g'''(f)
\]

\[
+ C_{4,2}(f', g') g''(f) + C_{4,1}(f', g') g'(f),
\]

\[
\ldots
\]

\[
f^{(n)}(g) = \left( \frac{f'}{g'} \right)^n g^{(n)}(f) + \left[ \frac{a_n(f')^{n-2} (f'' g' - f'' g'')}{(g')^{n+1}} \right] g^{(n-1)}(f)
\]

\[
+ C_{n,2}(f', g') g^{n-2}(f) + \cdots + C_{n,1}(f', g') g'(f),
\]

where \( C_{i,j}(f', g') (i \geq 3, 1 \leq j \leq i - 2) \) are rational functions of \( g', g'', \ldots, g^{(i-1)} \) and \( f', f'', \ldots, f^{(i-1)} \).
Replacing \( z \) by \( g(z) \) in Equation (2.1) yields

\[
(2.5) \quad p_n(g) f^{(n)}(g) + p_{n-1}(g) f^{(n-1)}(g) + \cdots + p_1(g) f'(g) + p_0(g) f(g) + p(g) = 0.
\]

Substituting (1.1) and (2.4) into (2.5) we deduce that

\[
(2.6) \quad p_n(g) \left( \frac{f'}{g} \right)^n g^{(n)}(f) + \left[ p_n(g) \frac{a_n(f')^{n-2}(f''g' - f'g'')}{(g')^{n+1}} + p_{n-1}(g) \left( \frac{f'}{g} \right)^{n-1} \right] g^{(n-1)}(f)
\]

\[+ D_{n,n-2}(f', g') g^{(n-2)}(f) + \cdots + D_{n,1}(f', g') g'(f) + p_0(g) g(f) + p(g) = 0,
\]

where \( D_{n,i}(f', g') \) for \( n \geq 3, 1 \leq i \leq n - 2 \) are rational functions of \( g, g', g'', \ldots, g^{(n)} \) and \( f', f'', \ldots, f^{(n)} \).

**Claim 2.8.** Let \( \{r_j\}_{j=1}^{\infty} \) tending to \( \infty \) be the sequence of positive numbers in Lemma 2.5. Then there exists a positive number \( K \) such that, for sufficiently large \( j \),

\[
T \left( r_j, p_n(g) \left( \frac{f'}{g} \right)^n \right) \leq KT(r_j, f),
\]

\[
T \left( r_j, p_n(g) \frac{a_n(f')^{n-2}(f''g' - f'g'')}{(g')^{n+1}} + p_{n-1}(g) \left( \frac{f'}{g} \right)^{n-1} \right) \leq KT(r_j, f) \quad \text{and}
\]

\[
T \left( r_j, D_{n,i}(f', g') \right) \leq KT(r_j, f)
\]

for all \( n \geq 3, 1 \leq i \leq n - 2 \).

**Proof of Claim 2.8.** We shall prove a more general result. Let \( P(f, f', \ldots, f^{(n)}, g, g', \ldots, g^{(n)}) \) be a linear combination of

\[
V(z) = b(z) f(z)^{s_0} f'(z)^{t_0} \cdots f^{(n)}(z)^{s_n} g(z)^{t_0} g'(z)^{t_1} \cdots g^{(n)}(z)^{t_n},
\]

where \( s_i, t_i \) (\( 0 \leq i \leq n \)) are integers and \( b(z) \) is a rational function. We shall prove that there exists a positive constant \( K \) such that, for all sufficiently large \( j \),

\[
T(r_j, P) \leq KT(r_j, f).
\]

In fact, by Nevanlinna’s Logarithmic Derivative Lemma, (see [8, Page 105]) we have \( T(r, f') \leq T(r, f) + O(\log r) \). Then, for \( 0 \leq i \leq n \),

\[
T(r_j, f^{(i)}) \leq T(r_j, f) + O(\log r).
\]
Since $\rho(f) > 0$, \(\liminf_{r\to\infty} (\log T(r, f)/\log r) = \rho(f) > 0\). Thus, for sufficiently large \(r\),

$\log r \leq \frac{\rho(f)}{2} \log T(r, f) = o(T(r, f)).$

Combining this with the above inequality implies that

\[ T(r_j, f^{(i)}) \leq 2T(r_j, f). \]

Now, by Lemma 2.5, there exists a positive constant \(K_1\) such that

\[ T(r_j, g^{(i)}) \leq K_1T(r_j, f) \quad \text{for } j \geq 1 \text{ and } 0 \leq i \leq n. \]

By Nevanlinna’s First Fundamental Theorem, \(T(r, 1/f) \leq 2T(r, f)\) for sufficiently large \(r\). Note that \(T(r, b) = o(T(r, f))\). Thus, from (2.7) and (2.8),

\[ T(r_j, V) \leq T(r_j, b) + \sum_{i=0}^{n} 2|s_i|T(r_j, f^{(i)}) + \sum_{i=0}^{n} 2|t_i|T(r_j, g^{(i)}) \leq K_2T(r_j, f) \]

for some positive constant \(K_2\) and for sufficiently large \(j\). Therefore, there exists a positive constant \(K\) such that

\[ T(r_j, P) \leq KT(r_j, f) \]

for sufficiently large \(j\). Claim 2.8 follows.

Now by (2.6), Claim 2.8 and Lemma 2.1, there exist \(n + 2\) polynomials \(Q_n(z), Q_{n-1}(z), \ldots, Q_1(z), Q_0(z)\) and \(Q(z)\), not all identically zero, such that

\[ Q_n(z)g^{(n)}(z) + Q_{n-1}(z)g^{(n-1)}(z) + \cdots + Q_0(z)g(z) + Q(z) = 0. \]

Substituting \(z\) by \(f(z)\) in this equation, we get

\[ Q_n(f)g^{(n)}(f) + Q_{n-1}(f)g^{(n-1)}(f) + \cdots + Q_0(f)g(f) + Q(f) = 0. \]

Eliminating the term \(g^{(n)}(f)\) from this and (2.6), we have

\[ H_{n-1}g^{(n-1)}(f) + H_{n-2}g^{(n-2)}(f) + \cdots + H_1g'(f) + H_0g(f) + H = 0, \]

where

\[
\begin{align*}
H_{n-1} &= Q_{n-1}(f)p_n(g) \left( \frac{f'}{g'} \right)^n - Q_n(f) \left[ p_n(g) \frac{a_n(f')^{n-2}(f''g' - f'g'')}{(g')^{n+1}} + p_{n-1}(g) \left( \frac{f'}{g'} \right)^{n-1} \right], \\
\cdots \\
H_1 &= Q_1(f)p_n(g) \left( \frac{f'}{g'} \right)^n - Q_0(f)p_0(g), \\
H &= Q(f)p_n(g) \left( \frac{f'}{g'} \right)^n - Q_n(f)p(g).
\end{align*}
\]
CLAIM 2.9. \( H_i \equiv 0 \) for \( 0 \leq i \leq n-1 \) and \( H \equiv 0 \).

PROOF OF CLAIM 2.9. Without loss of generality, we suppose on the contrary that \( H_{n-1} \not\equiv 0 \).

Then, from (1.1), (2.4) and (2.9) we deduce that
\[
H_{n-1} \left( \frac{g'}{f'} \right)^{n-1} f^{(n-1)}(g) + E_{n-1,1}(f', g') f^{(n-2)}(g) + \cdots + E_{n-1,n-2}(f', g') f'(g) + H_0 f(g) + H = 0,
\]
where \( E_{n-1,i}(f', g')(n \geq 3, 1 \leq i \leq n-2) \) are rational functions of \( g, g', g'', \ldots, g^{(n)} \) and \( f', f'', \ldots, f^{(n)} \). By Claim 2.8 and Lemma 2.1, there exist \( n+1 \) polynomials \( t_{n-1}(z), t_{n-2}(z), \ldots, t_1(z), t_0(z) \) and \( t(z) \), not all identically zero, such that
\[
t_{n-1}(z) f^{(n-1)}(z) + t_{n-2}(z) f^{(n-2)}(z) + \cdots + t_0(z) f(z) + t(z) = 0.
\]
This contradicts condition 3 of the lemma. Claim 2.9 follows.

Thus, we have
\[
Q_{n-1}(f) p_n(g) \left( \frac{f'}{g'} \right)^n - Q_n(f) \left[ p_n(g) \frac{a_n f^{n-2} (f'' g - f' g')}{g^{n+1}} + p_n-1(g) \left( \frac{f'}{g'} \right)^{n-1} \right] = 0,
\]
(2.10)
\[
Q_0(f) p_n(g) \left( \frac{f'}{g'} \right)^n - Q_n(f) p_0(g) = 0 \quad \text{and} \quad Q(f) p_n(g) \left( \frac{f'}{g'} \right)^n - Q_n(f) p(g) = 0.
\]

CLAIM 2.10. \( Q_n \not\equiv 0 \) and \( Q_0 \not\equiv 0 \).

PROOF OF CLAIM 2.10. In fact, if \( Q_n \equiv 0 \) then, by the same arguments as used in the proof of Claim 2.8, we can deduce that \( f(z) \) must be a transcendental entire solution of some differential equation with order at most \( n-1 \) and with polynomial coefficients. This is a contradiction. Hence \( Q_n \not\equiv 0 \). It follows from (2.10) that \( Q_0 \not\equiv 0 \). Claim 2.10 follows.

This completes the proof of Lemma 2.7.

3. Proof of Theorem 1.1

By (2.2) and (2.3), we have
\[
\frac{Q(f)}{Q_0(f)} = \frac{p(g)}{p_0(g)}.
\]
Let
\[ R_1(z) = \frac{Q(z)}{Q_0(z)}, \quad R_2(z) = \frac{p(z)}{p_0(z)}. \]

By assumption, \( p(z)/p_0(z) \) is not a constant. Therefore \( R_2(z) \) is not constant. Also \( R_1(z) \) is not constant by (3.1).

4. Proof of Theorem 1.2

Recall that
\[ f(z) = p(z) + p_1(z)e^{q_1(z)} + p_2(z)e^{q_2(z)} + \cdots + p_n(z)e^{q_n(z)}. \]

Without loss of generality, we assume that \( \deg q_1 \leq \deg q_2 \leq \cdots \leq \deg q_n \). Then \( \rho(f) = \lambda(f) = \max\{\deg q_1, \ldots, \deg q_n\} = \deg q_n \).

For \( 1 \leq i \leq n \) and \( 0 \leq j \leq n \), set
\[ u_{i,j} = \frac{(p_i e^{q_i})^j}{e^{q_i}}. \]

It is easy to see that all \( u_{i,j} \) are non-zero polynomials. Note from (4.1) that
\[ u_{i,j} = u_{i,j+1}. \]

for all \( j \geq 0 \) and \( 1 \leq i \leq n \). From
\[ f = p + p_1e^{q_1} + p_2e^{q_2} + \cdots + p_n e^{q_n} \]
we get
\[ f^{(j)} = p^{(j)} + u_{1,j}e^{q_1} + u_{2,j}e^{q_2} + \cdots + u_{n,j}e^{q_n}, \]
for all \( j \geq 0 \).

Set
\[ A = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \\ u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \end{bmatrix}, \quad R_0 = \begin{bmatrix} u_{1,n} & u_{2,n} & \cdots & u_{n,n} \\ u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \end{bmatrix}, \]
\[ B_1 = \begin{bmatrix} f - p & p_2 & \cdots & p_n \\ f' - p' & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ (f - p)^{(n-1)} & u_{2,n-1} & \cdots & u_{n,n-1} \end{bmatrix}. \]
It is easy to check that $A$ and $R_0$ are two polynomials with $A \not\equiv 0$ and $R_0 \not\equiv 0$. In fact, if $A \equiv 0$, that is
\[
\begin{vmatrix}
p_{1,1} \quad p_{2,1} \quad \cdots \quad f - p

u_{1,1} \quad u_{2,1} \quad \cdots \quad f' - p'

\vdots \quad \vdots \quad \ddots \quad \vdots

u_{1,n} \quad u_{2,n} \quad \cdots \quad (f - p)^{(n-1)}
\end{vmatrix} = 0,
\]
then
\[
\begin{vmatrix}
p_{1,1}e^{q_1} \quad p_{2,1}e^{q_2} \quad \cdots \quad p_{n}e^{q_n}

u_{1,1}e^{q_1} \quad u_{2,1}e^{q_2} \quad \cdots \quad u_{n,1}e^{q_n}

\vdots \quad \vdots \quad \ddots \quad \vdots

u_{1,n-1}e^{q_1} \quad u_{2,n-1}e^{q_2} \quad \cdots \quad u_{n,n-1}e^{q_n}
\end{vmatrix} \equiv 0.
\]
Now, by Lemma 2.4, the functions $p_{1,1}e^{q_1}, p_{2,1}e^{q_2}, \ldots, p_{n}e^{q_n}$ are linearly dependent. This obviously contradicts Lemma 2.2. Similarly we can show that $R_0 \not\equiv 0$.

Now from (4.3) and (4.4), we deduce that
\[
e^{q_s} = B_s/A \quad (s = 1, \ldots, n).
\]
Substituting these into (4.4) for $j = n$, we have
\[
A f^{(n)} = p^{(n)} + u_{1,n}B_1 + u_{2,n}B_2 + \cdots + u_{n,n}B_n.
\]
Note that each $B_i$ ($1 \leq i \leq n$) is a linear combination of $f, f', \ldots, f^{(n-1)}$. We deduce that
\[
A f^{(n)} + R_{n-1} f^{(n-1)} + \cdots + R_1 f' + R_0 f + R = 0,
\]
where $R_{n-1}, \ldots, R_0$ and $R$ are polynomials.

Further, $f$ cannot be a solution of a differential equation
\[
t_{n-1} f^{(n-1)} + t_{n-2} f^{(n-2)} + \cdots + t_1 f' + t_0 f + t = 0,
\]
where $t_{n-1}, \ldots, t_1, t_0$ and $t$ are polynomials, not all of them zero. For suppose to the contrary that $f(z)$ is a solution of (4.5). Then this, (4.3) and (4.4) (with $j = n$) imply that
\[
(u_{1,n-1}t_{n-1} + u_{1,n-2}t_{n-2} + \cdots + u_{1,1}t_1 + t_0 p_{1})e^{q_1} + \cdots
\]
\[
+ (u_{n,n-1}t_{n-1} + u_{n,n-2}t_{n-2} + \cdots + u_{n,1}t_1 + t_0 p_{n})e^{q_n}
\]
\[
+ (t_{n-1} p^{(n-1)} + t_{n-2} p^{(n-2)} + \cdots + t_1 p + t) \equiv 0.
\]
Combining this with Lemma 2.2, we get

\[ u_{1,n-1}t_{n-1} + u_{1,n-2}t_{n-2} + \cdots + u_{1,1}t_1 + t_0p_1 = 0 \]

\[ \vdots \]

\[ u_{n,n-1}t_{n-1} + u_{n,n-2}t_{n-2} + \cdots + u_{n,1}t_1 + t_0p_n = 0. \]

These obviously contradict \( A \not\equiv 0 \).

Thus the conditions of Lemma 2.7 are satisfied, so there exist polynomials \( Q(z) \), \( Q_0(z) \), \( Q_{n-1}(z) \) and \( Q_n(z) \), with \( Q_0(z) \not\equiv 0 \) and \( Q_n(z) \not\equiv 0 \), such that

\[ Q_{n-1}(f)A(g)\left( \frac{f'}{g'} \right)^n - Q_n(f)A(g)\left( \frac{f'}{g'} \right)^{n-1} + R_{n-1}(g)\left( \frac{f'}{g'} \right)^{n-1} = 0, \]

\[ Q_0(f)A(g)\left( \frac{f'}{g'} \right)^n - Q_n(f)R_0(g) = 0 \quad \text{and} \]

\[ Q(f)A(g)\left( \frac{f'}{g'} \right)^n - Q_n(f)R(g) = 0. \]

We remark that

\[ R_{n-1} = -u_{1,n}(-1)^{n+1} \begin{vmatrix} p_2 & p_3 & \cdots & p_n \\ u_{2,1} & u_{2,2} & \cdots & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1,2} & u_{n-1,3} & \cdots & u_{n-1,n} \end{vmatrix} - u_{2,n}(-1)^{n+2} \begin{vmatrix} p_1 & p_3 & \cdots & p_n \\ u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-2,1} & u_{n-2,2} & \cdots & u_{n-2,n} \end{vmatrix} - \cdots - u_{n,n}(-1)^{n+n} \begin{vmatrix} p_1 & p_2 & \cdots & p_{n-1} \\ u_{1,1} & u_{1,2} & \cdots & u_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-2,1} & u_{n-2,2} & \cdots & u_{n-2,n} \end{vmatrix} \]

\[ = - \begin{vmatrix} p_1 & p_2 & \cdots & p_n \\ u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-2} & u_{2,n-2} & \cdots & u_{n,n-2} \end{vmatrix} - u_{1,n}(-1)^{n+1} \begin{vmatrix} p_2 & p_3 & \cdots & p_n \\ u_{2,1} & u_{2,2} & \cdots & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1,1} & u_{n-1,2} & \cdots & u_{n-1,n} \end{vmatrix}. \]
Similarly, for any \( i \) with \( 1 \leq i \leq n - 1 \), one has

\[
R_{n-i} = \begin{vmatrix}
  p_1 & p_2 & \cdots & p_n \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{1,n-i-1} & u_{2,n-i-1} & \cdots & u_{n,n-i-1} \\
  u_{1,n-i+1} & u_{2,n-i+1} & \cdots & u_{n,n-i+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{1,n} & u_{2,n} & \cdots & u_{n,n}
\end{vmatrix}.
\]

Also, it is easy to verify that

\[
R = p R_0 + p' R_1 + \cdots + p^{(n-1)} R_{n-1} - p^n A,
\]

with

\[
\deg R_0 \geq \max \{ \deg R_i \mid 1 \leq i \leq n - 1 \}, \quad \deg A.
\]

Here we only prove that \( \deg R_0 \geq \deg A \). In fact, we have

\[
\deg R_0 = \deg A + \sum_{i=1}^{n} (\deg q_i - 1).
\]

Set

\[
Z = \begin{vmatrix}
  u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \\
  u_{1,n} & u_{2,n} & \cdots & u_{n,n}
\end{vmatrix}.
\]

To establish (4.8), we need only prove that \( \deg Z = \deg A + \sum_{i=1}^{n} (\deg q_i - 1) \). Rewrite \( Z \) by (4.2) as \( Z = Z_{11} + Z_{12} \) where

\[
Z_{11} = \begin{vmatrix}
  u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \\
  u'_{1,n-1} & u'_{2,n-1} & \cdots & u'_{n,n-1}
\end{vmatrix},
\]

\[
Z_{12} = \begin{vmatrix}
  u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \\
  q_1 u_{1,n-1} & q_2 u_{2,n-1} & \cdots & q_n u_{n,n-1}
\end{vmatrix}.
\]

We easily deduce that \( \deg Z = \deg Z_{12} \). Now we decompose \( Z_{12} = Z_{121} + Z_{122} \).
where

\[
Z_{121} = \begin{vmatrix}
  u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{1,n-2} & u_{2,n-2} & \cdots & u_{n,n-2} \\
  q'_{1}u_{1,n-1} & q'_{2}u_{2,n-1} & \cdots & q'_{n}u_{n,n-1}
\end{vmatrix},
\]

\[
Z_{122} = \begin{vmatrix}
  u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\
  \vdots & \vdots & \ddots & \vdots \\
  q'_{1}u_{1,n-2} & q'_{2}u_{2,n-2} & \cdots & q'_{n}u_{n,n-2} \\
  q'_{1}u_{1,n-1} & q'_{2}u_{2,n-1} & \cdots & q'_{n}u_{n,n-1}
\end{vmatrix},
\]

We easily deduce that \( \deg Z_{12} = \deg Z_{122} \). Thus \( \deg Z = \deg Z_{122} \). This procedure can be repeated. Finally we can assert that \( \deg Z = \deg Z_{12-2} \), where

\[
Z_{12-2} = \begin{vmatrix}
  q'_{1}p_{1} & q'_{2}p_{2} & \cdots & q'_{n}p_{n} \\
  \vdots & \vdots & \ddots & \vdots \\
  q'_{1}u_{1,n-2} & q'_{2}u_{2,n-2} & \cdots & q'_{n}u_{n,n-2} \\
  q'_{1}u_{1,n-1} & q'_{2}u_{2,n-1} & \cdots & q'_{n}u_{n,n-1}
\end{vmatrix} = q'_{1}q'_{2} \cdots q'_{n}A,
\]

so establishing (4.8).

From (4.6) it follows that \( p \equiv \text{constant} \) implies \( R/R_{0} \equiv \text{constant} \). We now prove the converse. Let us suppose that

\[
\frac{R}{R_{0}} = c.
\]

By (4.6),

\[
(c - p)R_{0} = p'R_{1} + \cdots + p^{(n-1)}R_{n-1} - p^{(n)}A.
\]

If \( p \not\equiv \text{constant} \), this contradicts (4.7).

Finally, the theorem follows from Lemma 2.7 and Theorem 1.1.

5. Proof of Corollary 1.3

By (2.2) and (2.3), we have

\[
(5.1) \quad \frac{Q(f)}{Q_{0}(f)} = \frac{p(g)}{p_{0}(g)}.
\]

Let

\[
R_{1}(z) = \frac{Q(z)}{Q_{0}(z)}, \quad R_{2}(z) = \frac{p(z)}{p_{0}(z)}.
\]
By assumption, \( p(z)/p_0(z) \) is not a constant, therefore, \( R_2(z) \) is not constant. Also \( R_1(z) \) is not constant by (5.1).

We rewrite (5.1) in the form

\[
Q_0(x)p(y) - Q(x)p_0(y) = 0
\]

and consider two subcases.

If \( Q_0(x)p(y) - Q(x)p_0(y) \equiv \) constant, then by (5.2)

\[
Q_0(x)p(y) - Q(x)p_0(y) \equiv 0.
\]

Hence

\[
\frac{Q(z)}{Q_0(z)} = \frac{p(z)}{p_0(z)} = S(z)
\]

for some rational function \( S(z) \). It follows from (5.1) that

\[
S(f) = S(g).
\]

Therefore \( f = \pm g + c \) for some constant \( c \). By Baker [1], we obtain \( J(f) = J(g) \).

If \( Q_0(x)p(y) - Q(x)p_0(y) \not\equiv \) constant, then by (3.1) we get a nonconstant polynomial \( Q_0(x)p(y) - Q(x)p_0(y) \) such that

\[
Q_0(f)p(g) - Q(f)p_0(g) \equiv 0.
\]

The conclusion follows from Lemma 2.6.

References