

STABLE PERTURBATION IN BANACH ALGEBRAS

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(Received 19 March 2005; revised 22 September 2006)

Communicated by A. Pryde

Abstract

Let \mathcal{A} be a unital Banach algebra. Assume that a has a generalized inverse a^+ . Then $\bar{a} = a + \delta a \in \mathcal{A}$ is said to be a stable perturbation of a if $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$. In this paper we give various conditions for stable perturbation of a generalized invertible element and show that the equation $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$ is closely related to the gap function $\hat{\delta}(\bar{a}\mathcal{A}, a\mathcal{A})$. These results will be applied to error estimates for perturbations of the Moore–Penrose inverse in C^* -algebras and the Drazin inverse in Banach algebras.

2000 *Mathematics subject classification*: primary 46H99, 46N40; secondary 65J05.

1. Introduction

Throughout the paper, \mathcal{A} is a complex Banach algebra with unit 1. Let $\text{GL}(\mathcal{A})$ denote the group of all invertible elements in \mathcal{A} . An element a in \mathcal{A} is said to be *generalized invertible* if there is a $b \in \mathcal{A}$ such that $aba = b$ and $bab = b$. Such an element b is called a *generalized inverse* of a , denoted by a^+ (certainly such an a^+ is not unique). If \mathcal{A} is a C^* -algebra and $a \in \mathcal{A}$, the *Moore–Penrose inverse* of a is defined as the element a^\dagger satisfying

$$(1.1) \quad aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (a^\dagger a)^* = a^\dagger a, \quad (aa^\dagger)^* = aa^\dagger.$$

When $a \neq 0$, a^\dagger is unique by [11]. We denote by $\text{GI}(\mathcal{A})$ the set of all generalized invertible elements in \mathcal{A} . If \mathcal{A} is a C^* -algebra and $a \in \text{GI}(\mathcal{A})$ then, by [11, Theorem 6], a^\dagger exists so that the set of all Moore–Penrose invertible elements in \mathcal{A}

is $\text{GI}(\mathcal{A})$. Recall from [9] that $a \in \mathcal{A}$ is *Drazin invertible* if there are a $b \in \mathcal{A}$ and a natural number k such that

$$(1.2) \quad a^k b a = a^k, \quad b a b = b, \quad a b = b a.$$

The least k such that (1.2) holds for some b is called the *index* of a , denoted by $\text{Ind}(a)$. In this case, the b in (1.2) is called the *Drazin inverse* of a and we denote it by a^D . When $\text{Ind}(a) = 1$, a^D is called the *group inverse* of a and we use the symbol $a^\#$ to denote it; if $a \neq 0$ and $\text{Ind}(a) = 0$ then $a \in \text{GL}(\mathcal{A})$. Put $\text{D}(\mathcal{A}) = \{a \in \mathcal{A} \mid a^D \text{ exists}\}$ and $\text{G}(\mathcal{A}) = \{a \in \text{D}(\mathcal{A}) \mid \text{Ind}(a) \leq 1\}$. It may be observed that $\text{GL}(\mathcal{A}) \subset \text{G}(\mathcal{A}) \subset \text{D}(\mathcal{A})$ and $\text{GL}(\mathcal{A}) \cup \text{G}(\mathcal{A}) \subset \text{GI}(\mathcal{A})$. Moreover, for any $a \in \text{D}(\mathcal{A})$ and $x \in \text{GL}(\mathcal{A})$,

$$x a x^{-1} \in \text{D}(\mathcal{A}), \quad \text{Ind}(x a x^{-1}) = \text{Ind}(a), \quad (x a x^{-1})^D = x a^D x^{-1}.$$

In recent years, many results have been published concerning the continuity of the Moore–Penrose inverse in C^* -algebras and the Drazin inverse in Banach algebras (see, for example, [10, 12, 14, 16]). In [10] and [16], Rakočević gave many equivalent conditions for the continuity of the Moore–Penrose inverse in C^* -algebras and the Drazin inverse in Banach algebras respectively. Connected with the continuity of generalized inverses and Drazin inverses, quantitative analysis of perturbations of Moore–Penrose inverses in C^* -algebras and Drazin inverses in Banach algebras has not been fully developed though Castro–González and Koliha in [2] and Rakočević and Wei in [17] have made a start on this programme.

Compared those with the study of the Moore–Penrose inverses on Hilbert spaces and Drazin inverses on Banach spaces, there are many fruitful results concerning the quantitative analysis of the perturbation of the Moore–Penrose inverses on Hilbert spaces and Drazin inverses on Banach spaces. For example, in [5, 7, 20] the author gave an estimate of perturbation bounds for the Moore–Penrose inverse on Hilbert spaces under stable perturbation of operators, which is a generalization of rank-preserving perturbation of matrices. Meanwhile, for the Drazin inverse on Banach spaces, many perturbation analysis results have been obtained in [2], [3], [4] and [13] by means of the gap between operators (which is the gap between their graphs) or the gap between the subspaces $\text{Ran}(T^k)$, $\text{Ran}(\bar{T}^k)$ and $\text{Ker } T^k$, $\text{Ker } \bar{T}^k$ of the Drazin invertible operators T and \bar{T} (where $k = \max\{\text{Ind}(T), \text{Ind}(\bar{T})\}$).

In order to give a quantitative analysis of perturbations of generalized inverses in C^* -algebras and the Drazin inverse in Banach algebras without using the gap function, we first generalize the concept of the so-called stable perturbation of operators in [6] to the case of Banach algebras and establish a self-contained perturbation theory. This theory provides several useful conditions for stable perturbation. Some of these involve deep properties of idempotents in Banach algebras. Then we apply this theory

to estimate the perturbation bounds of the Moore–Penrose inverse in C^* -algebras and the Drazin inverse in Banach algebras.

2. Stable perturbation in Banach algebras

Let X be a Banach space and $B(X)$ be the Banach algebra of all bounded linear operators on X . For $T \in B(X)$, we write $\text{Ran}(T)$ (respectively $\text{Ker } T$) to denote the range (respectively null space) of T . Let $T \in \text{GI}(B(X))$ and $\bar{T} = T + \delta T \in B(X)$. We say that \bar{T} is a *stable perturbation* of T if $\text{Ran}(\bar{T}) \cap \text{ker } T^+ = \{0\}$ for some generalized inverse T^+ of T (cf. [6, Definition 3.1]). When \bar{T} is a stable perturbation of T and $\|T^+\| \|\delta T\| < 1$, \bar{T} has a generalized inverse of the form $\bar{T}^+ = (I + T^+\delta T)^{-1}T^+$ (cf. [6, Corollary 3.1]). It is also noted that if \bar{T} and T are $m \times n$ matrices with $\|T^+\| \|\delta T\| < 1$, then $\text{rank } \bar{T} = \text{rank } T$ if and only if $\text{Ran}(\bar{T}) \cap \text{Ker } T^+ = \{0\}$ (cf. [6, Corollary 3.2]). In short, from [5, 6, 7, 20] we can see that this concept plays a very important role in studying perturbations of generalized inverses in infinite-dimensional spaces.

For any $a \in \mathcal{A}$, let L_a be the left multiplier on \mathcal{A} , so that $L_ax = ax$ for all $x \in \mathcal{A}$. Then $L_a \in B(\mathcal{A})$ and $\text{Ran}(L_a) = a\mathcal{A}$ and $\text{Ker } L_a = \{x \in \mathcal{A} | ax = 0\}$. It is easy to check that if $a \in \text{GI}(\mathcal{A})$ then $L_a \in \text{GI}(B(\mathcal{A}))$ and L_{a^+} is one of its generalized inverses. Moreover, $\text{Ker}(L_{a^+}) = (1 - aa^+)\mathcal{A}$. Thus, if $\bar{a} = a + \delta a \in \mathcal{A}$ then

$$\text{Ran}(L_{\bar{a}}) \cap \text{Ker}(L_{a^+}) = \bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A}.$$

This leads to the following definition.

DEFINITION 2.1. Let \mathcal{A} be a unital Banach algebra and $\bar{a} = a + \delta a \in \mathcal{A}$ for $a \in \text{GI}(\mathcal{A})$. We say that \bar{a} is a *stable perturbation* of a (with respect to a^+) if $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$.

The following conditions provide means for efficient handling of stable perturbations of elements of $\text{GI}(\mathcal{A})$.

PROPOSITION 2.2. Let $a \in \text{GI}(\mathcal{A})$ and $\bar{a} = a + \delta a \in \mathcal{A}$ with $\|a^+\| \|\delta a\| < 1$. Then the following conditions are equivalent.

- (1) $\bar{a} \in \text{GI}(\mathcal{A})$ with $\bar{a}^+ = (1 + a^+\delta a)^{-1}a^+$;
- (2) \bar{a} is a stable perturbation of a ;
- (3) $\bar{a}(1 + a^+\delta a)^{-1}(1 - a^+a) = 0$;
- (4) $(1 - aa^+)(1 + \delta aa^+)^{-1}\bar{a} = 0$;
- (5) $(1 - aa^+)\delta a(1 - a^+a) = (1 - aa^+)\delta a(1 + a^+\delta a)^{-1}a^+\delta a(1 - a^+a)$.

PROOF. (1) \Rightarrow (2). Let $x \in \bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A}$. Then $a^+x = 0$ and $x = \bar{a}y$ for some $y \in \mathcal{A}$. Put $z = (1 + \delta aa^+)x$. Using $a^+(1 + \delta aa^+)^{-1} = (1 + a^+\delta a)^{-1}a^+$, we have

$$\bar{a}\bar{a}^+ = (a + \delta a)a^+(1 + \delta aa^+)^{-1} = 1 - (1 - aa^+)(1 + \delta aa^+)^{-1}.$$

Thus $(1 - \bar{a}\bar{a}^+)z = x = \bar{a}y$ and hence $x = 0$.

(2) \Rightarrow (3). Set $z = \bar{a}(1 + a^+\delta a)^{-1}(1 - a^+a) \in \bar{a}\mathcal{A}$. Since

$$aa^+z = a(1 + a^+\delta a)(1 + a^+\delta a)^{-1}(1 - a^+a) = 0,$$

it follows that $z \in \bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$, so $z = 0$.

(3) \Leftrightarrow (4). In fact:

$$\begin{aligned} (1 - aa^+)(1 + \delta aa^+)^{-1}\bar{a} &= (1 + \delta aa^+ - \bar{a}a^+)(1 + \delta aa^+)^{-1}\bar{a} \\ &= \bar{a} - \bar{a}(1 + a^+\delta a)^{-1}(a^+\delta a + 1 + a^+a - 1) \\ &= \bar{a}(1 + a^+\delta a)^{-1}(1 - a^+a). \end{aligned}$$

(3) \Rightarrow (5). We have

$$\begin{aligned} (1 - aa^+)\delta a(1 + a^+\delta a)^{-1}a^+\delta a(1 - a^+a) \\ = (1 - aa^+)\delta a(1 - a^+a) - (1 - aa^+)\delta a(1 + a^+\delta a)^{-1}(1 - a^+a) \\ = (1 - aa^+)\delta a(1 - a^+a). \end{aligned}$$

(5) \Rightarrow (1). The computation above shows that if (5) is true then

$$(1 - aa^+)\delta a(1 + a^+\delta a)^{-1}(1 - a^+a) = 0.$$

Since $aa^+\bar{a}(1 + a^+\delta a)^{-1}(1 - a^+a) = 0$, we have $\bar{a}(1 + a^+\delta a)^{-1}(1 - a^+a) = 0$. Put $b = (1 + a^+\delta a)^{-1}a^+$. Then $\bar{a}b\bar{a} = \bar{a}$ and $b\bar{a}b = b$, that is, $\bar{a} \in \text{GI}(\mathcal{A})$ with $\bar{a}^+ = (1 + a^+\delta a)^{-1}a^+$. \square

COROLLARY 2.3. *Let \mathcal{B} be a unital Banach subalgebra of \mathcal{A} . Let $a \in \text{GI}(\mathcal{B})$ and $\bar{a} = a + \delta a \in \mathcal{B}$ with $\|a^+\| \|\delta a\| < 1$. Then $\bar{a}\mathcal{B} \cap (1 - aa^+)\mathcal{B} = \{0\}$ if and only if $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$.*

PROOF. The “if” part is obvious. We now prove the “only if” part. By Proposition 2.2 (1), $\bar{a}\mathcal{B} \cap (1 - aa^+)\mathcal{B} = \{0\}$ implies that $\bar{a}^+ = (1 + a^+\delta a)^{-1}a^+$ for $\bar{a} \in \text{GI}(\mathcal{B})$. Since $\text{GI}(\mathcal{B}) \subset \text{GI}(\mathcal{A})$, we have $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$ by using Proposition 2.2 again. \square

Let V_1, V_2 be subspaces in X . Put

$$\delta(V_1, V_2) = \begin{cases} 0, & V_1 = \{0\} \\ \sup \{ \text{dist}(x, V_2) \mid x \in V_1, \|x\| = 1 \}, & V_1 \neq \{0\}. \end{cases}$$

The gap function $\hat{\delta}(V_1, V_2)$ of V_1 and V_2 is defined by

$$\hat{\delta}(V_1, V_2) = \max \{ \delta(V_1, V_2), \delta(V_2, V_1) \}.$$

By [8, Lemma 3.2], $\delta(V_1, V_2) = \delta(\overline{V}_1, \overline{V}_2)$, where \overline{V}_i is the closure of V_i in X for $i = 1, 2$.

LEMMA 2.4. *For idempotents $p_1, p_2 \in \mathcal{A}$, $\hat{\delta}(p_1\mathcal{A}, p_2\mathcal{A}) \leq \|p_1 - p_2\|$.*

PROOF. The assertion is trivial when $p_1 = 0$. If $p_1 \neq 0$ then for any $z \in p_1\mathcal{A}$ with $\|z\| = 1$,

$$\text{dist}(z, p_2\mathcal{A}) \leq \|p_1z - p_2z\| \leq \|p_1 - p_2\|.$$

Thus $\delta(p_1\mathcal{A}, p_2\mathcal{A}) \leq \|p_1 - p_2\|$ and hence $\hat{\delta}(p_1\mathcal{A}, p_2\mathcal{A}) \leq \|p_1 - p_2\|$. \square

PROPOSITION 2.5. *Let $a \in \text{GI}(\mathcal{A})$ and $\bar{a} = a + \delta a \in \mathcal{A}$ with $\|a^+\| \|\delta a\| < 1$.*

- (1) *If $aa^+ \neq 1$ and $\delta(\bar{a}\mathcal{A}, a\mathcal{A}) < 1/\|1 - aa^+\|$ then $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$;*
- (2) *If $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$ then*

$$\hat{\delta}(\bar{a}\mathcal{A}, a\mathcal{A}) \leq \frac{\|1 - aa^+\| \|a^+\| \|\delta a\|}{1 - \|a^+\| \|\delta a\|}.$$

PROOF. (1) If $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} \neq \{0\}$, we can find $x \in \bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A}$ with $\|x\| = 1$. Then for any $y \in \mathcal{A}$, $(1 - aa^+)(x - ay) = x$ and it follows that $1 \leq \|1 - aa^+\| \|x - ay\|$. Thus $\delta(\bar{a}\mathcal{A}, a\mathcal{A}) \geq \|1 - aa^+\|^{-1}$, which contradicts the assumption.

(2) In this case, $\bar{a} \in \text{GI}(\mathcal{A})$ with $\bar{a} = (1 + a^+\delta a)^{-1}a^+$ by Proposition 2.2. Note that $\bar{a}\bar{a}^+\mathcal{A} = \bar{a}\mathcal{A}$ and $aa^+\mathcal{A} = a\mathcal{A}$. Therefore, by Lemma 2.4,

$$\begin{aligned} \hat{\delta}(\bar{a}\mathcal{A}, a\mathcal{A}) &\leq \|\bar{a}\bar{a}^+ - aa^+\| = \|(1 - aa^+)[1 - (1 + a^+\delta a)^{-1}]\| \\ &\leq \frac{\|1 - aa^+\| \|a^+\| \|\delta a\|}{1 - \|a^+\| \|\delta a\|}. \end{aligned} \quad \square$$

REMARK. If $\|a^+\| \|\delta a\| < (1 + \|1 - aa^+\|^2)^{-1}$ then Proposition 2.5 shows that $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$ if and only if $\hat{\delta}(\bar{a}\mathcal{A}, a\mathcal{A}) < \|1 - aa^+\|^{-1}$.

As an application of stable perturbation in C^* -algebras, we give perturbation analysis for generalized inverses in a C^* -algebra as follows.

PROPOSITION 2.6. *Let \mathcal{A} be a unital C^* -algebra. Suppose $a \in \text{GI}(\mathcal{A})$ and $\bar{a} = a + \delta a \in \mathcal{A}$ with $\|a^\dagger\| \|\delta a\| < 1$. Suppose that $\bar{a}\mathcal{A} \cap (1 - aa^\dagger)\mathcal{A} = \{0\}$. Then $\bar{a} \in \text{GI}(\mathcal{A})$ and*

$$(2.1) \quad \|\bar{a}^\dagger\| \leq \frac{\|a^\dagger\|}{1 - \|a^\dagger\| \|\delta a\|}, \quad \frac{\|\bar{a}^\dagger - a^\dagger\|}{\|a^\dagger\|} \leq \frac{1 + \sqrt{5}}{2} \frac{\|a^\dagger\| \|\delta a\|}{1 - \|a^\dagger\| \|\delta a\|}.$$

PROOF. By [15], \mathcal{A} has a faithful representation (π, H_π) such that $\pi(\mathcal{A})$ is a unital C^* -subalgebra of $B(H)$ with $I = \pi(1)$. Put $A = \pi(a)$, $\bar{A} = \pi(\bar{a})$, $\delta A = \bar{A} - A$. Then $\bar{A}[B(H_\pi)] \cap (I - AA^\dagger)[B(H_\pi)] = \{0\}$ by Corollary 2.3. Let $\eta \in \text{Ran}(\bar{A}) \cap \text{Ker} A^\dagger$. Then $A^\dagger \eta = 0$ and $\eta = \bar{A}\xi_0$ for some $\xi_0 \in H_\pi$. Choose a nonzero vector $\xi_1 \in H_\pi$ and put $A_1\xi = (\xi, \xi_1)\xi_0$, $A_2\xi = (\xi, \xi_1)\eta$ for all $\xi \in H_\pi$. Then $A_2 = \bar{A}A_1$ and $A_2 = (I - AA^\dagger)A_2$. Thus

$$A_2 \in \bar{A}[B(H_\pi)] \cap (I - AA^\dagger)[B(H_\pi)] = \{0\}$$

so that $\eta = 0$, that is, \bar{A} is a stable perturbation of A as operators on H_π . Therefore \bar{A}^\dagger exists and $\|\bar{A}^\dagger\| \leq \|A^\dagger\|/(1 - \|A^\dagger\|\|\delta A\|)$ by [7, Theorem 1] and

$$\frac{\|\bar{A}^\dagger - A^\dagger\|}{\|A^\dagger\|} \leq \frac{1 + \sqrt{5}}{2} \frac{\|A^\dagger\| \|\delta A\|}{1 - \|A^\dagger\| \|\delta A\|}$$

by [20, Proposition 7].

Since π is a $*$ -isometry and 0 is an isolated point of the spectrum of $\bar{A}^*\bar{A}$, it follows that 0 is an isolated point of the spectrum of $\bar{a}^*\bar{a}$, that is, $\bar{a} \in \text{GI}(\mathcal{A})$ and (2.1) follows. □

Let \mathcal{A} be a unital C^* -algebra and $\{a_n\}_0^\infty \subset \text{GI}(\mathcal{A}) \setminus \{0\}$ with $\lim_{n \rightarrow \infty} a_n = a_0$. In [16], Rakočević gave various equivalent conditions that make $\lim_{n \rightarrow \infty} a_n^\dagger = a_0^\dagger$ in \mathcal{A} . Combining [16, Theorem 2.2] with Lemma 2.4 and Proposition 2.6, we have the following result.

COROLLARY 2.7. *Let \mathcal{A} be a unital C^* -algebra and $\{a_n\}_0^\infty \subset \text{GI}(\mathcal{A}) \setminus \{0\}$ with $\lim_{n \rightarrow \infty} a_n = a_0$. Then following conditions are equivalent:*

- (1) $a_n^\dagger \rightarrow a_0^\dagger$;
- (2) $a_n a_n^\dagger \rightarrow a_0 a_0^\dagger$;
- (3) $a_n^\dagger a_n \rightarrow a_0^\dagger a_0$;
- (4) $\sup_n \|a_n^\dagger\| < +\infty$;
- (5) $a_n \mathcal{A} \cap (1 - a_0 a_0^\dagger) \mathcal{A} = \{0\}$ for n large enough.

PROOF. The proof of the equivalence of (1), (2), (3) and (4) follows from [16, Theorem 2.2].

(3) \Rightarrow (5). By Lemma 2.4,

$$\hat{\delta}(a_n \mathcal{A}, a_0 \mathcal{A}) = \hat{\delta}(a_n a_n^\dagger \mathcal{A}, a_0 a_0^\dagger \mathcal{A}) \leq \|a_n a_n^\dagger - a_0 a_0^\dagger\|.$$

So $\lim_{n \rightarrow \infty} \hat{\delta}(a_n \mathcal{A}, a_0 \mathcal{A}) = 0$ and hence, $a_n \mathcal{A} \cap (1 - a_0 a_0^\dagger) \mathcal{A} = \{0\}$ for n large enough by Proposition 2.5 (take $\delta a_0 = a_n - a_0$ and $\bar{a}_0 = a_0 + \delta a_0$).

(5) \Rightarrow (1). When $a_n \mathcal{A} \cap (1 - a_0 a_0^\dagger) \mathcal{A} = \{0\}$ for sufficiently large n , we have $\lim_{n \rightarrow \infty} a_n^\dagger = a_0^\dagger$ by Proposition 2.6. □

3. Some further conditions for stable perturbation

We know from [6, Corollary 3.1] that, for $T, \bar{T} = T + \delta T \in B(X)$ with T^+ existing and $\|T^+\| \|\delta T\| < 1$, if $\dim \text{Ker } \bar{T} = \dim \text{Ker } T < +\infty$ (or $\text{rank } \bar{T} = \text{rank } T$ when $\dim X < +\infty$) then \bar{T} is a stable perturbation of T . We will extend this result to the case of Banach algebras.

For $a \in \mathcal{A}$, put $\text{Ker}(a) = \{x \in \mathcal{A} | ax = 0\}$ and $\text{coKer}(a) = \{x \in \mathcal{A} | xa = 0\}$. Obviously, if $a \in \text{GI}(\mathcal{A})$ then $\text{Ker}(a) = (1 - a^+a)\mathcal{A}$ and $\text{coKer}(a) = \mathcal{A}(1 - aa^+)$.

LEMMA 3.1. *Let $a \in \text{GI}(\mathcal{A})$ and $\bar{a} = a + \delta a \in \mathcal{A}$ with $\|a^+\| \|\delta a\| < 1$. Put*

$$p_a = (1 + a^+\delta a)^{-1}(1 - a^+a), \quad q_a = (1 - aa^+)(1 + \delta aa^+)^{-1}.$$

Then $\text{Ker}(\bar{a}) \subset p_a\mathcal{A}$ and $\text{coKer}(\bar{a}) \subset \mathcal{A}q_a$.

PROOF. Let $x \in \text{Ker}(\bar{a})$ and $y \in \text{coKer}(\bar{a})$. Then

$$(a^+a - 1 + 1 + a^+\delta a)x = 0, \quad y(aa^+ - 1 + 1 + \delta aa^+) = 0.$$

Consequently, $x = p_ax, y = yq_a$, that is, $\text{Ker}(\bar{a}) \subset p_a\mathcal{A}$ and $\text{coKer}(\bar{a}) \subset \mathcal{A}q_a$. \square

Let p, q be idempotents in a complex Banach algebra \mathcal{E} . Recall that p and q are equivalent in \mathcal{E} (in symbols, $p \sim q$) if there are $x, y \in \mathcal{E}$ such that $p = xy, q = yx$. Note that x, y can be chosen so that $px = xq = x$ and $qy = yp = y$. Also p and q are said to be similar (in symbols, $p \approx q$), if there is an $a \in \text{GL}(\mathcal{E})$ such that $p = a^{-1}qa$ when \mathcal{E} is unital; if \mathcal{E} is nonunital $p \approx q$ means that there is an $a \in \text{GL}(\tilde{\mathcal{E}})$ such that $p = a^{-1}qa$, where $\tilde{\mathcal{E}} = \{\lambda + x | \lambda \in \mathbb{C}, x \in \mathcal{E}\}$.

Clearly, if $p \approx q$, then $p \sim q$.

Let \mathcal{E} be a Banach algebra with unit 1. \mathcal{E} is said to be finite if for any idempotent $e \in \mathcal{E}$ with $e \sim 1$, we have $e = 1$. If \mathcal{E} is nonunital and $\tilde{\mathcal{E}}$ is finite, we will say \mathcal{E} is finite. For example, every finite dimensional Banach algebra is finite; the matrix algebra over a commutative Banach algebra is finite; and the Banach algebra $\mathcal{K}(X)$ consisting of all compact operators on X is finite (this can be shown by means of the Fredholm index).

Let p be a nonzero idempotent in \mathcal{E} . We will say p is finite if $p\mathcal{E}p$ is finite.

LEMMA 3.2. *Let p, q, r be nonzero idempotents in a Banach algebra \mathcal{E} .*

- (1) *If $p \sim q$ and $q \sim r$ then $p \sim r$;*
- (2) *If p is finite and $pq = qp = q$ and $p \sim q$ then $p = q$;*
- (3) *If p is finite and $p \sim q$ then q is finite.*

PROOF. (1) Let $x_1, y_1, x_2, y_2 \in \mathcal{E}$ be such that

$$p = x_1y_1, \quad q = y_1x_1, \quad q = x_2y_2, \quad r = y_2x_2.$$

Put $z_1 = x_1x_2, z_2 = y_2y_1$. Then $z_1z_2 = p, z_2z_1 = r$.

(2) Let $x, y \in \mathcal{E}$ be such that

$$p = xy, \quad q = yx, \quad px = xq = x, \quad qy = yp = y.$$

Since $pq = qp = q$, it follows that $x = pxp, y = pyp$. Thus $p \sim q$ in $p^\mathcal{E}p$. Noting that p is the unit of $p^\mathcal{E}p$ and $p^\mathcal{E}p$ is finite, we have $p = q$.

(3) Let e be an idempotent in $q^\mathcal{E}q$ with $e \sim q$. Let $a, b \in \mathcal{E}$ be such that

$$(3.1) \quad p = ab, \quad q = ba, \quad pa = aq = a, \quad qb = bp = b.$$

Put $f = aeb$. Then $fp = pf = f$ by (3.1). From $eq = qe = e$, we get that $f^2 = f$. Since $f \sim e \sim q \sim p$ by (1) and p is finite, we have $f = p$ by (2) and hence $e = q$ by (3.1). \square

THEOREM 3.3. Let \mathcal{A} be a unital Banach algebra and let $a, \bar{a} = a + \delta a \in \text{GI}(\mathcal{A})$ with $\|a^+\| \|\delta a\| < 1$. If $1 - a^+a$ (or $1 - aa^+$) is finite and $1 - a^+a \sim 1 - \bar{a}^+\bar{a}$ (or $1 - aa^+ \sim 1 - \bar{a}\bar{a}^+$), then $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$.

PROOF. Let p_a, q_a be as in Lemma 3.1. From

$$p_a = (1 + a^+\delta a)^{-1}(1 - a^+a)(1 + a^+\delta a), \quad q_a = (1 + \delta aa^+)(1 - aa^+)(1 + \delta aa^+)^{-1},$$

we see that p_a and q_a are idempotents and $p_a \approx 1 - a^+a, q_a \approx 1 - aa^+$.

Now put $\bar{p} = 1 - \bar{a}^+\bar{a}, \bar{q} = 1 - \bar{a}\bar{a}^+$. Then, by Lemma 3.1, $p_a\bar{p} = \bar{p}$ and $\bar{q}q_a = \bar{q}$. Set $w_1 = 1 + \bar{p}p_a(1 - \bar{p}), w_2 = 1 + (1 - \bar{q})q_a\bar{q}$. Then $w_1, w_2 \in \text{GL}(\mathcal{A})$ with $w_1^{-1} = 1 - \bar{p}p_a(1 - \bar{p}), w_2^{-1} = 1 - (1 - \bar{q})q_a\bar{q}$. Set $\bar{p}_1 = w_1p_a w_1^{-1}, \bar{q}_1 = w_2^{-1}q_a w_2$. Using $p_a\bar{p} = \bar{p}, \bar{q}q_a = \bar{q}$, we obtain

$$(3.2) \quad \bar{p}_1 = p_a - \bar{p}p_a(1 - \bar{p}), \quad \bar{p}_1\bar{p} = \bar{p}\bar{p}_1 = \bar{p};$$

$$(3.3) \quad \bar{q}_1 = q_a - (1 - \bar{q})q_a\bar{q}, \quad \bar{q}_1\bar{q} = \bar{q}\bar{q}_1 = \bar{q}.$$

Since $1 - a^+a \sim \bar{p}_1 \sim \bar{p}$ (or $1 - aa^+ \sim \bar{q}_1 \sim \bar{q}$) and $1 - a^+a$ (or $1 - aa^+$) is finite, it follows from (3.1) (or (3.2)) and Lemma 3.2 that $\bar{p}_1 = \bar{p}$ (or $\bar{q}_1 = \bar{q}$), that is, $p = \bar{p}p$ (or $q = \bar{q}\bar{q}$) and hence

$$\bar{a}(1 + a^+\delta a)^{-1}(1 - a^+a) = 0 \quad (\text{or } (1 - aa^+)(1 + \delta aa^+)^{-1}\bar{a} = 0).$$

Therefore $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$ by Proposition 2.2. \square

Let p be a nonzero idempotent in Banach algebra \mathcal{E} . We say p is minimal if $p\mathcal{E}p = \{\lambda p | \lambda \in \mathbb{C}\}$. Let $M(\mathcal{E})$ denote the set of all minimal idempotents in \mathcal{E} . Now suppose that \mathcal{E} is a semiprime Banach algebra (that is, for any $x \in \mathcal{E} \setminus \{0\}$ there is $y \in \mathcal{E}$ such that $xyx \neq 0$) and $M(\mathcal{E}) \neq \emptyset$. Then the socle $S_{\mathcal{E}}$ is defined as the smallest ideal which contains $\{\mathcal{E}p | p \in M(\mathcal{E})\}$ (or $\{p\mathcal{E} | p \in M(\mathcal{E})\}$) (cf. [18, pages 45–47]).

According to [1], a right (or left) ideal M of \mathcal{E} is of finite order if M is the sum of a finite number of minimal right (or left) ideals of \mathcal{E} . The order $\theta(M)$ of M is defined as the smallest number of minimal right (or left) ideals which have sum M when $M \neq \{0\}$ and $\theta(M) = 0$ if $M = \{0\}$. By [1, Lemma 1.1], $\theta(M) = m \geq 1$ if and only if there are $e_1, \dots, e_m \in M(\mathcal{E}) \cap M$ with $e_i e_j = 0, i \neq j, i, j = 1, \dots, m$ such that $M = (\sum_{i=1}^m e_i)\mathcal{E}$ (or $\mathcal{E}(\sum_{i=1}^m e_i)$).

THEOREM 3.4. *Let \mathcal{A} be a unital Banach algebra with $M(\mathcal{A}) \neq \emptyset$. Let $a \in \text{GI}(\mathcal{A})$ and $\bar{a} = a + \delta a \in \mathcal{A}$ with $\|a^+\| \|\delta a\| < 1$. Assume that $\text{Ker}(a)$ (or $\text{coKer}(a)$) is of finite order. Then so is the $\text{Ker}(\bar{a})$ (or $\text{coKer}(\bar{a})$).*

In addition, if $\theta(\text{Ker}(\bar{a})) = \theta(\text{Ker}(a))$ (or $\theta(\text{coKer}(\bar{a})) = \theta(\text{coKer}(a))$) then \bar{a} is stable perturbation of a .

PROOF. We only give the proof for right ideals $\text{Ker}(a)$ and $\text{Ker}(\bar{a})$. The proof of the remainder is similar.

Let $p_a = (1 + a^+\delta a)^{-1}(1 - a^+a)$. Since $p_a \approx 1 - a^+a$, we have

$$\theta(p_a\mathcal{A}) = \theta((1 - a^+a)\mathcal{A}) = \theta(\text{Ker}(a)) < +\infty.$$

Suppose that $\theta(\text{Ker}(a)) = n \geq 1$. Then there are $e_1, \dots, e_n \in M(\mathcal{A}) \cap p_a\mathcal{A}$ such that

$$(3.4) \quad p_a\mathcal{A} = e_1\mathcal{A} + \dots + e_n\mathcal{A}, \quad e_i e_j = 0, \quad i \neq j, \quad i, j = 1, \dots, n.$$

Put $e = \sum_{i=1}^n e_i$. Then $p_a\mathcal{A} = e\mathcal{A}$, that is, $p_a e = e, e p_a = p_a$. Since $\text{Ker}(\bar{a}) \subset p_a\mathcal{A}$ by Lemma 3.1, we have

$$e[\text{Ker}(\bar{a})] = \text{Ker}(\bar{a}) \quad \text{and by (3.4)} \quad e_i[\text{Ker}(\bar{a})] \subset e_i\mathcal{A}, \quad i = 1, \dots, n.$$

Assume that $\text{Ker}(\bar{a}) \neq \{0\}$. Since both $e_i[\text{Ker}(\bar{a})]$ and $e_i\mathcal{A}$ are right ideals and $e_i\mathcal{A}$ is minimal, it follows that

$$e_i[\text{Ker}(\bar{a})] = \{0\} \quad \text{or} \quad e_i[\text{Ker}(\bar{a})] = e_i\mathcal{A}, \quad i = 1, \dots, n.$$

Without loss of generality, we may assume that $e_i[\text{Ker}(\bar{a})] = e_i\mathcal{A}$ for $i = 1, \dots, k$ and $e_i[\text{Ker}(\bar{a})] = \{0\}$ for $i = k + 1, \dots, n$. Put $e' = \sum_{i=1}^k e_i$. Then

$$\text{Ker}(\bar{a}) = e[\text{Ker}(\bar{a})] = e'[\text{Ker}(\bar{a})] = e'\mathcal{A}.$$

This means that $\theta(\text{Ker}(\bar{a})) = k \leq n$.

If $\theta(\text{Ker}(\bar{a})) = \theta(\text{Ker}(a))$, we have $\text{Ker}(\bar{a}) = e_{\mathcal{A}} = p_a \mathcal{A}$ by the above argument. Thus $\bar{a}(1 + a^+ \delta a)^{-1}(1 - a^+ a) = 0$ and hence $\bar{a} \mathcal{A} \cap (1 - aa^+) \mathcal{A} = \{0\}$ by Proposition 2.2. □

4. Perturbation analysis for Drazin inverse

Let $a \in G(\mathcal{A})$ and put $a^\pi = 1 - aa^\#$. Then a^π is an idempotent and $aa^\pi = a^\pi a = 0$ and $a^\# a^\pi = a^\pi a^\# = 0$. Let $\bar{a} = a + \delta a \in \mathcal{A}$ with $\kappa_\#(a)\epsilon_a < (1 + \|a^\pi\|)^{-1}$, where $\kappa_\#(a) = \|a\| \|a^\#\|$ and $\epsilon_a = \|\delta a\| \|a\|^{-1}$. Put

$$\Phi(a) = 1 + \delta a (1 - aa^\#) \delta a \left[(I + a^\# \delta a)^{-1} a^\# \right]^2.$$

Since

$$\|1 - \Phi(a)\| \leq \|a^\pi\| \left(\frac{\kappa_\#(a)\epsilon_a}{1 - \kappa_\#(a)\epsilon_a} \right)^2 < 1,$$

it follows that $\Phi(a) \in \text{GL}(\mathcal{A})$ and $\Phi^{-1}(a) = (\Phi(a))^{-1} = \sum_{n=0}^\infty (1 - \Phi(a))^n$ with

$$(4.1) \quad \|\Phi^{-1}(a)\| \leq \frac{(1 - \kappa_\#(a)\epsilon_a)^2}{(1 - \kappa_\#(a)\epsilon_a)^2 - \|a^\pi\| (\kappa_\#(a)\epsilon_a)^2}.$$

LEMMA 4.1. *Let $a \in G(\mathcal{A})$ and $\bar{a} = a + \delta a \in \mathcal{A}$ with $\kappa_\#(a)\epsilon_a < (1 + \|a^\pi\|)^{-1}$.*

Put

$$C(a) = a^\pi \delta a (1 + a^\# \delta a)^{-1} a^\#, \quad D(a) = (1 + a^\# \delta a)^{-1} a^\# \Phi^{-1}(a).$$

If $\bar{a} \cap (1 - aa^\#) \mathcal{A} = \{0\}$ then $\bar{a} \in G(\mathcal{A})$ with

$$(4.2) \quad \bar{a}^\# = (1 + C(a))(D(a) + D^2(a)\delta aa^\pi)(1 - C(a)).$$

PROOF. Clearly, $1 + C(a) \in \text{GL}(\mathcal{A})$ with $(1 + C(a))^{-1} = 1 - C(a)$. Applying conditions (3) and (5) from Proposition 2.2 to $\bar{a}_0 = (1 - C(a))\bar{a}(1 + C(a))$, we have

$$\begin{aligned} \bar{a}_0 &= a + aa^\# \delta a a a^\# + aa^\# \delta a a^\pi \delta a (1 + a^\# \delta a) a^\# + aa^\# \delta a a^\pi \\ &= aa^\# \Phi(a) a (1 + a^\# \delta a) aa^\# + aa^\# \delta a a^\pi. \end{aligned}$$

Noting that $a^\pi (1 + a^\# \delta a)^{-1} a^\# = 0$ and $\Phi(a) a^\pi = a^\pi$, it can be checked that

$$\begin{aligned} \bar{a}_0^\# &= (1 + a^\# \delta a)^{-1} a^\# \Phi^{-1}(a) + [(1 + a^\# \delta a)^{-1} a^\# \Phi^{-1}(a)]^2 \delta a a^\pi \\ &= D(a) + D^2(a)\delta aa^\pi. \end{aligned}$$

Therefore $\bar{a}^\# = (1 + C(a))(D(a) + D^2(a)\delta aa^\pi)(1 - C(a))$. □

We now give a perturbation analysis for group inverses under stable perturbation as follows.

THEOREM 4.2. *Let $a \in G(\mathcal{A})$ and $\bar{a} = a + \delta a \in \mathcal{A}$ with $\kappa_{\#}(a)\epsilon_a < (1 + \|a^{\pi}\|)^{-1}$. Assume that $\bar{a}\mathcal{A} \cap (1 - a\bar{a}^{\#})\mathcal{A} = \{0\}$. Then $\bar{a} \in G(\mathcal{A})$ and*

$$\|\bar{a}^{\#}\| \leq \frac{\|a^{\#}\|}{[1 - (1 + \|a^{\pi}\|)\kappa_{\#}(a)\epsilon_a]^2}, \quad \frac{\|\bar{a}^{\#} - a^{\#}\|}{\|a^{\#}\|} \leq \frac{(1 + 2\|a^{\pi}\|)\kappa_{\#}(a)\epsilon_a}{[1 - (1 + \|a^{\pi}\|)\kappa_{\#}(a)\epsilon_a]^2}.$$

PROOF. We keep $C(a)$, $D(a)$ as in Lemma 4.1. Then, by Lemma 4.1, $\bar{a} \in G(\mathcal{A})$. Since $D(a)(I - C(a)) = D(a)$, it follows from (4.2) that

$$(4.3) \quad \bar{a}^{\#} = (I + C(a))D(a) + (I + C(a))D^2(a)\delta a a^{\pi}(I - C(a)),$$

$$(4.4) \quad \|\bar{a}^{\#}\| \leq (1 + \|C(a)\|)\|D(a)\| + (1 + \|C(a)\|)^2\|D(a)\|^2\|\delta a\|\|a^{\pi}\|.$$

We have, by (4.1),

$$\|D(a)\| \leq \frac{(1 - \kappa_{\#}(a)\epsilon_a)\|a^{\#}\|}{[1 - (1 + \|a^{\pi}\|)\kappa_{\#}(a)\epsilon_a][1 + (\|a^{\pi}\| - 1)\kappa_{\#}(a)\epsilon_a]}.$$

Since

$$1 + \|C(a)\| \leq \frac{1 + (\|a^{\pi}\| - 1)\kappa_{\#}(a)\epsilon_a}{1 - \kappa_{\#}(a)\epsilon_a},$$

we deduce that

$$(4.5) \quad \|\bar{a}^{\#}\| \leq \frac{(1 - \kappa_{\#}(a)\epsilon_a)\|a^{\#}\|}{[1 - (1 + \|a^{\pi}\|)\kappa_{\#}(a)\epsilon_a]^2} \leq \frac{\|a^{\#}\|}{[1 - (1 + \|a^{\pi}\|)\kappa_{\#}(a)\epsilon_a]^2}$$

by (4.4). Finally, by (4.3),

$$\|\bar{a}^{\#} - a^{\#}\| \leq \|D(a) - a^{\#}\| + \|C(a)D(a)\| + (1 + \|C(a)\|)^2\|D(a)\|^2\|\delta a\|\|a^{\pi}\|.$$

Now we have

$$\begin{aligned} \|D(a) - a^{\#}\| &= \|(1 + a^{\pi}\delta a)^{-1}a^{\#} - a^{\#}\| \Phi^{-1}(a) + a^{\#}(\Phi^{-1}(a) - 1)\| \\ &\leq \frac{\|a^{\#}\|\kappa_{\#}(a)\epsilon_a}{1 - \kappa_{\#}(a)\epsilon_a} \|\Phi^{-1}(a)\| + \|a^{\#}\| \|\Phi^{-1}(a)\| \|\Phi(a) - 1\| \\ &\leq \frac{\|a^{\#}\|\kappa_{\#}(a)\epsilon_a}{1 - (1 + \|a^{\pi}\|)\kappa_{\#}(a)\epsilon_a}, \\ \|C(a)D(a)\| &\leq \frac{\|a^{\pi}\|\kappa_{\#}(a)\epsilon_a}{1 - \kappa_{\#}(a)\epsilon_a} \frac{(1 - \kappa_{\#}(a)\epsilon_a)\|a^{\#}\|}{1 - 2\kappa_{\#}(a)\epsilon_a - (\|a^{\pi}\| - 1)(\kappa_{\#}(a)\epsilon_a)^2} \\ &\leq \frac{\|a^{\#}\|\|a^{\pi}\|\kappa_{\#}(a)\epsilon_a}{1 - (1 + \|a^{\pi}\|)\kappa_{\#}(a)\epsilon_a} \quad ((\kappa_{\#}(a)\epsilon_a)^2 \leq \kappa_{\#}(a)\epsilon_a < 1). \end{aligned}$$

Therefore

$$\|\bar{a}^{\#} - a^{\#}\| \leq \frac{(1 + 2\|a^{\pi}\|)\|a^{\#}\|\kappa_{\#}(a)\epsilon_a}{[1 - (1 + \|a^{\pi}\|)\kappa_{\#}(a)\epsilon_a]^2}.$$

□

Let $a, \bar{a} = a + \delta a \in \mathcal{A}$ and put $\delta a^j = (a + \delta a)^j - a^j, j = 1, \dots, n$. Then

$$\|\delta a^j\| \leq (\|a\| + \|\delta a\|)^{j-1} \|\delta a\| + \|a\| \|\delta a^{j-1}\|.$$

Suppose $\epsilon_a < 1$. From above, we can deduce that

$$(4.6) \quad \|\delta a^n\| \leq \|\delta a\| \sum_{j=0}^{n-1} \|a\|^{n-1} (1 + \epsilon_a)^{n-1-j} < (2^n - 1) \|a\|^n \epsilon_a.$$

Let $a \in D(\mathcal{A})$ with $\text{Ind}(a) = k$. Put $a^\pi = 1 - aa^D, \kappa_D(a) = \|a\| \|a^D\|$. It is well-known that $(a^n)^\# = (a^D)^n, a^n (a^D)^n = 1 - a^\pi$ and $a^D = (a^n)^\# a^{n-1}$ for any $n \geq k$. So $\text{Ker}(a^n) = a^\pi \mathcal{A}$ and $a^n \mathcal{A} = (1 - a^\pi) \mathcal{A}$ for all $n \geq k$.

Using Theorem 4.2, we can give perturbation bounds for Drazin invertible elements in a Banach algebra \mathcal{A} as follows.

COROLLARY 4.3. *Let $a, \bar{a} = a + \delta a \in D(\mathcal{A})$ with $\text{Ind}(a) = k_1, \text{Ind}(\bar{a}) = k_2$ and*

$$\kappa_D^n(a) \epsilon_a < \frac{1}{(2^n - 1)(1 + \|a^\pi\|)}, \quad \text{where } n = \max\{k_1, k_2\}.$$

If $(1 - \bar{a}^\pi) \mathcal{A} \cap a^\pi \mathcal{A} = \{0\}$ then

$$\begin{aligned} \|\bar{a}^D\| &\leq \frac{2^{n-1} \kappa_D^{n-1}(a) \|a^D\|}{[1 - (2^n - 1)(1 + \|a^\pi\|) \kappa_D^n(a) \epsilon_a]^2}, \\ \frac{\|\bar{a}^D - a^D\|}{\|a^D\|} &\leq \frac{2^{n-1} (2^n - 1)(1 + 2 \|a^\pi\|) \kappa_D^{2n-1}(a) \epsilon_a}{[1 - (2^n - 1)(1 + \|a^\pi\|) \kappa_D^n(a) \epsilon_a]^2} + (2^{n-1} - 1) \kappa_D^{n-1}(a) \epsilon_a. \end{aligned}$$

PROOF. Noting that $\kappa_D(a) \geq \|aa^D\| \geq 1$, we have $\epsilon_a < 1$ and

$$\|\delta a^n\| \|(a^n)^\#\| < (2^n - 1) \kappa_D^n(a) \epsilon_a < \frac{1}{1 + \|a^\pi\|}$$

by (4.6). Since $\bar{a}^D = (\bar{a}^n)^\# \bar{a}^{n-1}, a^D = (a^n)^\# a^{n-1}$, it follows that

$$\begin{aligned} \|\bar{a}^D\| &\leq \|(\bar{a}^n)^\#\| \|a\|^{n-1} (1 + \epsilon_a)^{n-1} < 2^{n-1} \|a\|^{n-1} \|(\bar{a}^n)^\#\|, \\ \|\bar{a}^D - a^D\| &\leq \|(\bar{a}^n)^\# - (a^n)^\#\| \|\bar{a}\|^{n-1} + \|(a^n)^\#\| \|\delta a^{n-1}\| \\ &< 2^{n-1} \|(\bar{a}^n)^\# - (a^n)^\#\| \|a\|^{n-1} + (2^{n-1} - 1) \|a^D\| \kappa_D^{n-1}(a) \epsilon_a \end{aligned}$$

by (4.6). Then applying Theorem 4.2 to $(a^n)^\#$ and $(\bar{a}^n)^\#$, we obtain the assertion. \square

Let $\{a_n\}_{n=0}^\infty \subset D(\mathcal{A})$ and $\lim_{n \rightarrow \infty} a_n = a_0$. By [10, Theorem 4.1], $\lim_{n \rightarrow \infty} a_n^D = a_0^D$ if and only if $\lim_{n \rightarrow \infty} a_n^D a_n = a_0^D a_0$, if and only if $\sup_{n \geq 1} \|a_n^D\| < +\infty$, etc. In addition, if $\sup_{n \geq 0} \text{Ind}(a_n) < +\infty$, we have the following.

COROLLARY 4.4. *Let $a_n \in D(\mathcal{A})$ for $n \geq 0$ with $\lim_{n \rightarrow \infty} a_n = a_0$ and suppose that $l = \sup_{n \geq 0} \text{Ind}(a_n) < +\infty$.*

- (1) $\lim_{n \rightarrow \infty} a_n^D = a_0^D$ if and only if $(1 - a_n^\pi)\mathcal{A} \cap a_0^\pi\mathcal{A} = \{0\}$ for n large enough.
- (2) If a_0^π is a finite idempotent in \mathcal{A} then $\lim_{n \rightarrow \infty} a_n^D = a_0^D$ if and only if $a_n^\pi \sim a_0^\pi$ for n large enough.
- (3) If $M(\mathcal{A}) \neq \phi$ and $\theta(a_0^\pi\mathcal{A}) < +\infty$ then $\lim_{n \rightarrow \infty} a_n^D = a_0^D$ if and only if $\theta(a_n^\pi\mathcal{A}) = \theta(a_0^\pi\mathcal{A})$ for sufficiently large n .

PROOF. (1) The “if” part comes from Corollary 4.3. On the other hand, since $a_n^l\mathcal{A} = (1 - a_n^\pi)\mathcal{A}$ for $n \geq 0$ and $\lim_{n \rightarrow \infty} a_n^D = a_0^D$, it follows from Lemma 2.4 that $\lim_{n \rightarrow \infty} \hat{\delta}(a_n^l\mathcal{A}, a_0^l\mathcal{A}) = 0$ and consequently $a_n^l\mathcal{A} \cap (1 - a_0^l(a_0^l)^\#)\mathcal{A} = \{0\}$ for n large enough by Proposition 2.5, that is, $(1 - a_n^\pi)\mathcal{A} \cap a_0^\pi\mathcal{A} = \{0\}$.

(2) If $a_n^\pi \sim a_0^\pi$ for n large enough then a_n^l is a stable perturbation of a_0^l by Theorem 3.3. Thus $\lim_{n \rightarrow \infty} a_n^D = a_0^D$ by Corollary 4.3.

On the other hand, if $\lim_{n \rightarrow \infty} a_n^D = a_0^D$ then for sufficiently large n we have $\|a_n^\pi - a_0^\pi\| < \|2a_0^\pi - 1\|^{-1}$. Put $z_n = 1 + (2a_0^\pi - 1)(a_n^\pi - a_0^\pi)$. Then $\|1 - z_n\| < 1$, that is, $z_n \in \text{GL}(\mathcal{A})$ and $z_n a_n^\pi z_n^{-1} = a_0^\pi$.

(3) If $\theta(a_n^\pi\mathcal{A}) = \theta(a_0^\pi\mathcal{A})$ for n large enough then $\lim_{n \rightarrow \infty} a_n^D = a_0^D$ by Theorem 3.4 and Corollary 4.3. Conversely, when $\lim_{n \rightarrow \infty} a_n^D = a_0^D$ we have $a_n^\pi \approx a_0^\pi$ for sufficiently large n by the proof of (2) and hence $\theta(a_n^\pi\mathcal{A}) = \theta(a_0^\pi\mathcal{A})$. □

REMARK. Corollary 4.4 (1) covers [10, Corollary 3.4] and Corollary 4.4 (2), (3) covers [19, Theorem 2].

Acknowledgements

The author would like to thank the referee for his (or her) kindly pointing out many errors in the paper.

Part of this work was carried out while the author was visiting the Mathematical Institute, University of Munster. He wishes to express his gratitude to Professor J. Cuntz for his hospitality and his helpful suggestions.

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