

GENERALIZED GAUSSIAN ESTIMATES AND RIESZ MEANS OF SCHRÖDINGER GROUPS

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Abstract

We show that generalized Gaussian estimates for selfadjoint semigroups $(e^{-tA})_{t \in \mathbb{R}_+}$ on L_2 imply L_p -boundedness of Riesz means and other regularizations of the Schrödinger group $(e^{itA})_{t \in \mathbb{R}}$. This generalizes results restricted to semigroups with a heat kernel, which are due to Sjöstrand, Alexopoulos and more recently Carron, Coulhon and Ouhabaz. This generalization is crucial for elliptic operators A that are of higher order or have singular lower order terms since, in general, their semigroups fail to have a heat kernel.

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Introduction

It is well known that the Schrödinger group $(e^{it\Delta})_{t \in \mathbb{R}}$ acts on $L_p(\mathbb{R}^D)$ only if $p = 2$ [18]. Various authors showed that suitable regularizations of the Schrödinger group such as the Riesz means

$$t^{-\alpha} \int_0^t (t-s)^{\alpha-1} e^{is\Delta} ds$$

act even on $L_p(\mathbb{R}^D)$ for $p \neq 2$; see, for example, the works of Lanconelli [20] on boundedness of $(I - \Delta)^{-\alpha} e^{it\Delta}$ and of Sjöstrand [26] on Riesz means. These results were extended by Alexopoulos [1] to Laplacians Δ on Lie groups and Riemannian manifolds where the heat semigroup satisfies Gaussian estimates.

Carron, Coulhon and Ouhabaz [10] generalized this approach to arbitrary self-adjoint operators A on measured metric spaces (Ω, μ, d) of some dimension D , that

is, $|B(x, \lambda r)| \leq C\lambda^D |B(x, r)|$ for all $x \in \Omega$, $r > 0$, $\lambda \geq 1$, where $B(x, r)$ denotes the ball around x of radius r and $|B(x, r)|$ its volume. They showed L_p -boundedness of suitable regularizations of the Schrödinger group $(e^{itA})_{t \in \mathbb{R}}$ provided A satisfies Gaussian estimates (GEs), that is, the e^{-tA} have integral kernels $k_t(x, y)$ satisfying

$$(1) \quad |k_t(x, y)| \leq |B(x, r_t)|^{-1} g\left(\frac{d(x, y)}{r_t}\right) \quad \text{for all } x, y \in \Omega, t > 0.$$

Here the r_t are suitable positive radii and $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a suitable decay function. The central part of [10] was to deduce from the GE (1) the following $L_p \rightarrow L_p$ -norm estimate for the semigroup:

$$(2) \quad \|e^{-zA}\|_{p \rightarrow p} \leq C_\varepsilon \left(\frac{|z|}{\operatorname{Re} z}\right)^{D|1/2-1/p|+\varepsilon} \quad \text{for all } p \in [1, \infty], z \in \mathbb{C}_+,$$

where $\mathbb{C}_+ := \{z \in \mathbb{C}; \operatorname{Re} z > 0\}$. Then one can apply directly the following result on Riesz means and regularized groups due to El-Mennaoui [23] and Boyadzhiev and de Laubenfels [9]. Recall that if X is a Banach space and $S \in \mathfrak{L}(X)$ is injective, then a strongly continuous family $(W(t))_{t \in \mathbb{R}}$ in $\mathfrak{L}(X)$ is called an S -regularized group if $W(0) = S$ and $W(s)W(t) = SW(s+t)$ for all $s, t \in \mathbb{R}$. Its generator B is defined by $B = S^{-1}W'(0)$ with maximal domain; see, for example, [13] for details.

PROPOSITION A. *Let (Ω, μ) be a measure space, A a non-negative selfadjoint operator on $L_2(\Omega)$, $p \in [1, \infty]$, and $\alpha > \beta \geq 0$ such that $\|e^{-zA}\|_{p \rightarrow p} \leq C(|z|/\operatorname{Re} z)^\beta$ for all $z \in \mathbb{C}_+$.*

(a) *The following Riesz means $(I_\alpha(t))_{t \in \mathbb{R}}$ are uniformly bounded on $L_p(\Omega)$:*

$$I_\alpha(t) := \begin{cases} t^{-\alpha} \int_0^t (t-s)^{\alpha-1} e^{-isA} ds & t \geq 0, \\ I_\alpha(-t)^* & t < 0. \end{cases}$$

(b) *If (e^{-tA}) is bounded analytic of angle $\pi/2$ on $L_p(\Omega)$, then $W_\alpha(t) := (I+A)^{-\alpha} e^{itA}$ defines a $(I+A)^{-\alpha}$ -regularized group on $L_p(\Omega)$ with generator iA satisfying*

$$\|W_\alpha(t)\|_{p \rightarrow p} \leq C(1+|t|)^\alpha \quad \text{for all } t \in \mathbb{R}.$$

Unfortunately, there are many important operators A which do not satisfy GEs (in particular, [10] cannot be applied!). This occurs, for example, for elliptic operators A that are of higher order or have singular lower order terms [12, 21]. However, in many of these cases A still satisfies so-called generalized Gaussian estimates (GGEs); see [11, 25]. By this we mean an estimate of the following type:

$$(3) \quad \|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p_o \rightarrow p'_o} \leq |B(x, r_t)|^{1/p'_o - 1/p_o} g\left(\frac{d(x, y)}{r_t}\right)$$

for all $x, y \in \Omega, t > 0$, and for some $p_o \in [1, 2)$. The GGE (3) for $p_o = 1$ is equivalent to the GE (1) [5, Proposition 2.9]. The central part of this paper is to deduce from the GGE (3) the following generalization and slight improvement of the $L_p \rightarrow L_p$ -norm estimate (2) (Theorem 1.1 below), which improves a result of Davies [11]:

$$(4) \quad \|e^{-zA}\|_{p \rightarrow p} \leq C \left(\frac{|z|}{\operatorname{Re} z} \right)^{D|1/2-1/p|} \quad \text{for all } p \in [p_o, p'_o], z \in \mathbb{C}_+.$$

We obtain new results on regularizations of the Schrödinger group $(e^{itA})_{t \in \mathbb{R}}$, as before directly from Proposition A, and for the optimal range $p \in [p_o, p'_o], \alpha > D|1/2-1/p|$; see Theorem 1.3 below. We want to mention that, for the class of operators A satisfying the GGE (3), the interval $[p_o, p'_o]$ is, in general, optimal for the existence of the semigroup $(e^{-tA})_{t \in \mathbb{R}_+}$ on L_p [12], and the $\|e^{-zA}\|_{p \rightarrow p}$ -estimate (4) is optimal also [3].

A singular integral theory based on GGEs allows us to extend other L_2 -properties of A (above the boundedness of regularizations of $(e^{itA})_{t \in \mathbb{R}}$ considered in this paper) to L_p for $p \in (p_o, p'_o)$. We mention the properties of having maximal regularity [5], an H^∞ functional calculus [6] or Riesz transforms [7, 17]. In [4], this approach was applied to so-called 'spectral multipliers', which yields results of the type $F(A) \in \mathcal{L}(L_p), p \in (p_o, p'_o)$ for more general functions F than the $F_{\alpha,t}(x) := t^{-\alpha} \int_0^t (t-s)^{\alpha-1} e^{-isx} ds$ corresponding to our Riesz means (that is, $F_{\alpha,t}(A) = I_\alpha(t)$). The advantage of the method in the present paper is that it allows us to include the cases $p = p_o, p'_o$ and gives a direct approach for Schrödinger groups avoiding singular integral theory.

1. Main results

We begin with some basic notation and assumptions. For the rest of this paper, (Ω, μ, d) is a metric measure space. By $B(x, r)$ we denote balls in Ω and by $|B(x, r)|$ or $v_r(x)$ their volume. For integral operators T , we denote their integral kernel by $k_T(x, y)$, that is, $Tf(x) = \int k_T(x, y)f(y) d\mu(y)$.

1.1. Optimal estimates for $\|e^{-zA}\|_{p \rightarrow p}$ In order to deduce from Proposition A optimal L_p -boundedness results for regularizations of Schrödinger groups $(e^{itA})_{t \in \mathbb{R}}$, one needs optimal $\|e^{-zA}\|_{p \rightarrow p}$ -estimates. For the case of operators satisfying GEs, (almost) optimal $\|e^{-zA}\|_{p \rightarrow p}$ -estimates are obtained in [10, Theorem 4.3] by making tricky use of the identity

$$\|k_T(\cdot, y)\|_2^2 = k_{T^*T}(y, y).$$

Here we optimize and generalize this method to GGEs. The right substitute of the above kernel identity can be seen in the elementary norm identity

$$(5) \quad \|T\|_{p \rightarrow 2}^2 = \|T^*T\|_{p \rightarrow p'}.$$

This allows us to prove the following $\|e^{-zA}\|_{p \rightarrow p}$ -estimate. The proof of this estimate will be given at the end of this subsection.

THEOREM 1.1. *Let (Ω, μ, d) be a space of dimension D and $p \in [1, 2)$. Let A be a non-negative selfadjoint operator on $L_2(\Omega)$ such that*

$$\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p \rightarrow p'} \leq |B(x, r_t)|^{1/p' - 1/p} g\left(\frac{d(x, y)}{r_t}\right)$$

for all $x, y \in \Omega, t \in \mathbb{R}_+$, where $r_t = t^{1/m}$ and $g(s) = C \exp(-bs^{m/(m-1)})$ for some $m \geq 2$, and $C, b > 0$. Then $(e^{-tA})_{t \in \mathbb{R}_+}$ is bounded analytic of angle $\pi/2$ on $L_p(\Omega)$, and we have

$$\|e^{-zA}\|_{p \rightarrow p} \leq C \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(1/p-1/2)} \quad \text{for all } z \in \mathbb{C}_+.$$

REMARK 1.2. (a) The Laplacian $A = -\Delta$ on $\Omega = \mathbb{R}^D$ shows the optimality of our result [3].

(b) Duality and interpolation with $\|e^{-zA}\|_{2 \rightarrow 2} \leq 1$ yield that $(e^{-tA})_{t \in \mathbb{R}_+}$ is bounded analytic of angle $\pi/2$ on $L_q(\Omega)$ for all $q \in [p, p'], q \neq \infty$, and we have

$$\|e^{-zA}\|_{q \rightarrow q} \leq C \left(\frac{|z|}{\operatorname{Re} z}\right)^{D|1/2-1/q|} \quad \text{for all } q \in [p, p'], z \in \mathbb{C}_+.$$

(c) In [11], Davies verified the hypothesis of Theorem 1.1 for elliptic operators A of order $m \in 2\mathbb{N}$ on $\Omega = \mathbb{R}^D$ and for $p := (2D/(D+m)) \vee 1$, but he only obtained the following weaker conclusion, see [11, Theorems 20 and 25]:

$$\|e^{-zA}\|_{p \rightarrow p} \leq C \left(\frac{|z|}{\operatorname{Re} z}\right)^{2D(1/p-1/2)} \quad \text{for all } z \in \mathbb{C}_+.$$

(d) For the special case $p = 1$, our Theorem 1.1 is a slight improvement of [10, Theorem 4.3], where the following estimate is obtained for all $\varepsilon > 0$:

$$\|e^{-zA}\|_{1 \rightarrow 1} \leq C_\varepsilon \left(\frac{|z|}{\operatorname{Re} z}\right)^{D/2+\varepsilon} \quad \text{for all } z \in \mathbb{C}_+.$$

Deducing from the $L_p \rightarrow L_{p'}$ GGE in the hypothesis, an $L_p \rightarrow L_2$ GGE, and extending the latter to complex times are the main steps in the following short proof of Theorem 1.1.

PROOF OF THEOREM 1.1. We identify g and \tilde{g} , where $\tilde{g}(s) = \tilde{C} \exp(-\tilde{b}s^{m/(m-1)})$. By Proposition 3.1 (i) below, the $L_p \rightarrow L_{p'}$ GGE in the hypothesis implies the following $L_p \rightarrow L_2$ GGE:

$$\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p \rightarrow 2} \leq |B(x, r_t)|^{1/2-1/p} g\left(\frac{d(x, y)}{r_t}\right) \quad \text{for all } t \in \mathbb{R}_+.$$

By Theorem 2.1 below, the latter extends to complex times $z \in \mathbb{C}_+$ as follows:

$$\left\| \chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)} \right\|_{p \rightarrow 2} \leq |B(x, r_z)|^{1/2-1/p} \left(\frac{|z|}{\operatorname{Re} z} \right)^{D(1/p-1/2)} g \left(\frac{d(x, y)}{r_z} \right)$$

for all $z \in \mathbb{C}_+$, where $r_z = (\operatorname{Re} z)^{1/m-1}|z|$. This implies by Proposition 3.1 (ii) below for $R = (|z|/\operatorname{Re} z)^{-D(1/p-1/2)} e^{-zA}$:

$$\left\| e^{-zA} \right\|_{p \rightarrow p} \leq C_0 \left(\frac{|z|}{\operatorname{Re} z} \right)^{D(1/p-1/2)} \quad \text{for all } z \in \mathbb{C}_+.$$

It remains to show that $(e^{-zA})_{z \in \mathbb{C}_+}$ is strongly continuous in L_p on all strict subsectors of \mathbb{C}_+ . Arguing as in [24], but on L_p instead of L_1 , one obtains the strong continuity on subsectors from the previous L_p - L_p -estimate. \square

1.2. L_p -boundedness of regularizations of $(e^{itA})_{t \in \mathbb{R}}$ Theorem 1.1 allows us to apply our approach given in the introduction, that is, to verify the $L_p \rightarrow L_p$ norm estimate

$$\left\| e^{-zA} \right\|_{p \rightarrow p} \leq C \left(\frac{|z|}{\operatorname{Re} z} \right)^{D|1/2-1/p|} \quad \text{for all } p \in [p_o, p'_o], z \in \mathbb{C}_+$$

and to obtain L_p -boundedness of Riesz means and $(I + A)^{-\alpha}$ -regularizations of $(e^{itA})_{t \in \mathbb{R}}$ directly from Proposition A. This yields the following result.

THEOREM 1.3. *Let (Ω, μ, d) be a space of dimension D and $p_o \in [1, 2)$. Let A be a non-negative selfadjoint operator on $L_2(\Omega)$ such that*

$$\left\| \chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)} \right\|_{p_o \rightarrow p'_o} \leq |B(x, r_t)|^{1/p'_o-1/p_o} g \left(\frac{d(x, y)}{r_t} \right)$$

for all $x, y \in \Omega, t \in \mathbb{R}_+$, where $r_t = t^{1/m}$ and $g(s) = C \exp(-bs^{m/(m-1)})$ for some $m \geq 2$, and $C, b > 0$. Then we have for all $p \in [p_o, p'_o]$ and $\alpha > D|1/2-1/p|$:

(a) *The following Riesz means $(I_\alpha(t))_{t \in \mathbb{R}}$ are uniformly bounded on $L_p(\Omega)$:*

$$I_\alpha(t) := \begin{cases} t^{-\alpha} \int_0^t (t-s)^{\alpha-1} e^{isA} ds & t \geq 0, \\ I_\alpha(-t)^* & t < 0. \end{cases}$$

(b) *If $p \neq \infty$ then $W_\alpha(t) := (I + A)^{-\alpha} e^{itA}$ defines an $(I + A)^{-\alpha}$ -regularized group $(W_\alpha(t))_{t \in \mathbb{R}}$ on $L_p(\Omega)$, with generator iA satisfying $\|W_\alpha(t)\|_{p \rightarrow p} \leq C(1 + |t|)^\alpha$ for all $t \in \mathbb{R}$.*

REMARK 1.4. (a) For the special case $p_o = 1$ our Theorem 1.3 corresponds to Theorems 5.1–5.2 in [10].

- (b) The Laplacian $A = -\Delta$ on $\Omega = \mathbb{R}^D$ shows the optimality of our result [3].
- (c) In Remark 1.2 (c) above we already mentioned Davies' estimate [11]

$$\|e^{-zA}\|_{p \rightarrow p} \leq C \left(\frac{|z|}{\operatorname{Re} z} \right)^{2D|1/2-1/p|} \quad \text{for all } p \in [p_o, p'_o], z \in \mathbb{C}_+$$

for elliptic operators A of order $m \in 2\mathbb{N}$ on $\Omega = \mathbb{R}^D$ and for $p_o = (2D/(D+m)) \vee 1$. By Proposition A, this yields the conclusion of Theorem 1.3 for the range $\alpha > 2D|1/2 - 1/p|$. Our result yielding the optimal range $\alpha > D|1/2 - 1/p|$ is new.

PROOF OF THEOREM 1.3. The assertions follow directly from Theorem 1.1 (in the form of Remark 1.2 (b)) and Proposition A for $\beta = D|1/2 - 1/p|$. □

1.3. Examples In this subsection, we give some examples of elliptic operators A for which our Theorem 1.3 on regularizations of the Schrödinger group $(e^{itA})_{t \in \mathbb{R}}$ applies, that is, for which the following GGE holds:

$$(6) \quad \|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p_o \rightarrow p'_o} \leq |B(x, r_t)|^{1/p'_o - 1/p_o} C \exp\left(-b \left(\frac{d(x, y)}{r_t}\right)^{m/(m-1)}\right)$$

for all $x, y \in \Omega, t \in \mathbb{R}_+, r_t = t^{1/m}$ and for some $m \geq 2, p_o \in [1, 2)$.

1.3.1. Higher order operators with bounded coefficients on \mathbb{R}^D These operators A are given by forms $\mathfrak{a} : H^k(\mathbb{R}^D) \times H^k(\mathbb{R}^D) \rightarrow \mathbb{C}$ of the type

$$\mathfrak{a}(u, v) = \int_{\mathbb{R}^D} \sum_{|\alpha|=|\beta|=k} a_{\alpha\beta} \partial^\alpha u \overline{\partial^\beta v} dx,$$

where we assume $a_{\alpha,\beta} = \overline{a_{\beta,\alpha}} \in L_\infty(\mathbb{R}^D)$ for all α, β and Garding's inequality

$$\mathfrak{a}(u, u) \geq \delta \|\nabla^k u\|_2^2 \quad \text{for all } u \in H^k(\mathbb{R}^D),$$

for some $\delta > 0$ and $\|\nabla^k u\|_2^2 := \sum_{|\alpha|=k} \|\partial^\alpha u\|_2^2$. Then \mathfrak{a} is a closed symmetric form. The associated operator A is given by $u \in D(A)$, and $Au = g$ if and only if $u \in H^k$ and $\int g \overline{v} dx = \mathfrak{a}(u, v)$ for all $v \in H^k$.

In this situation, the GGE (6) holds for $p_o := (2D/(m+D)) \vee 1$ and $m := 2k$; see, for example, [11] and [2, Section 1.7]. Hence the conclusion of Theorem 1.3 holds for all $p \in [p_o, p'_o]$ and $\alpha > D|1/2 - 1/p|$.

1.3.2. Schrödinger operators with singular potentials on \mathbb{R}^D Now we study Schrödinger operators $A = -\Delta + V$ on $\mathbb{R}^D, D \geq 3$, where $V = V_+ - V_-, V_\pm \geq 0$

are locally integrable, and V_+ is bounded for simplicity (for the general case, see for example [25]). We assume the following form bound:

$$\int (V_-u)\bar{u} dx \leq \gamma (\|\nabla u\|_2^2 + \langle V_+u, u \rangle) + c(\gamma)\|u\|_2^2 \quad \text{for all } u \in H^1(\mathbb{R}^D)$$

and some $\gamma \in (0, 1)$. Then the form sum $A := -\Delta + V = (-\Delta + V_+) - V_-$ is defined and the associated form is closed and symmetric with form domain $H^1(\mathbb{R}^D)$. By standard arguments using ellipticity and the Sobolev inequality, the GGE (6) holds for $p_o = 2D/(D + 2)$ and $m = 2$ (after replacing A by $A + c(\gamma)$). Due to [21], $(e^{-tA})_{t \in \mathbb{R}_+}$ is bounded on $L_q(\mathbb{R}^D)$ for all $q \in (p_\gamma, p'_\gamma)$ and

$$p_\gamma := \frac{2D}{D(1 + \sqrt{1 - \gamma}) + 2(1 - \sqrt{1 - \gamma})} < \frac{2D}{D + 2}.$$

Hence, by interpolation, one obtains the GGE (6) even for all $p_o \in (p_\gamma, 2)$. Thus, the conclusion of Theorem 1.3 holds for all $p \in (p_\gamma, p'_\gamma)$ and $\alpha > D|1/2 - 1/p|$.

1.3.3. Elliptic operators on Riemannian manifolds Let $A = -\Delta$ be the Laplacian on a Riemannian manifold Ω . Let d be the geodesic distance and μ the Riemannian measure. Assume that Ω satisfies the so-called volume doubling property and that the heat kernel $k_t(x, y)$ satisfies

$$k_t(x, x) \leq C|B(x, \sqrt{t})|^{-1} \quad \text{for all } x \in \Omega, t > 0.$$

Then $(e^{tA})_{t \in \mathbb{R}_+}$ satisfies GEs [16] or, equivalently, the GGE (6) for $p_o = 1$ and $m = 2$. Hence the results we obtain on regularizations of $(e^{-itA})_{t \in \mathbb{R}}$ are contained in [10].

2. Extension of GGEs for $(e^{-tA})_{t \in \mathbb{R}_+}$ to $(e^{-zA})_{z \in \mathbb{C}_+}$

Theorem 1.3 on regularizations of Schrödinger groups is based on GGEs of the type

$$(7) \quad \|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p \rightarrow q} \leq |B(x, r_t)|^{1/q-1/p} g\left(\frac{d(x, y)}{r_t}\right).$$

Here we show, for analytic semigroups, how the latter estimate for real times t extends to an estimate for complex times z of the type

$$(8) \quad \|\chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)}\|_{p \rightarrow q} \leq |B(x, r_z)|^{1/q-1/p} C(z) g\left(\frac{d(x, y)}{r_z}\right).$$

This is important for the proof of our $\|e^{-zA}\|_{p \rightarrow p}$ -estimate in Theorem 1.1. Moreover, many other applications of GGEs require such complex time estimates, for example, the H^∞ functional calculus; see, for example, [6, 14].

In our setting, we have to extend (7) to the whole right halfplane \mathbb{C}_+ . This is crucial in order to obtain optimal results on Riesz means or more general so-called 'spectral multipliers' of selfadjoint semigroups; see [4, 10, 15].

For extensions of (7) to strict subsectors of the sector of analyticity of the semigroup, one can usually choose $r_z := r_{|z|}$ and $C(z) = 1$ in (8); see [19, 21]. The latter choice is impossible on the whole of \mathbb{C}_+ , as can be seen by fixing $x \neq y$ and by letting $|\arg z| \rightarrow \pi/2$.

The right candidate for the constant $C(z)$ in (8) is $C(z) = R(z)^{D(1/p-1/q)}$, where $R(z) := r_z/r_{\operatorname{Re}z}$. This is suggested by Proposition 3.1, Remark 3.2 below and confirmed by our following extension result. Its proof is given in Section 3.

We consider the standard case $r_t = t^{1/m}$ and $g(s) = Ce^{-bs^\omega}$, which corresponds (for $\omega = m/(m-1)$) to the typical estimates for elliptic operators A of order m ; see [2, 11].

THEOREM 2.1. *Let (Ω, μ, d) be a space of dimension D and $1 \leq p \leq p_o \leq q \leq \infty$. Let $(e^{-tA})_{t \in \mathbb{R}_+}$ be a semigroup of linear operators on $L_{p_o}(\Omega)$, which has a bounded and analytic extension to \mathbb{C}_+ and satisfies, for all $t \in \mathbb{R}_+$,*

$$\left\| \chi_{B(x, r_t)} e^{-tA} \chi_{B(y, r_t)} \right\|_{p \rightarrow q} \leq |B(x, r_t)|^{1/q-1/p} C e^{-b(d(x, y)/r_t)^\omega},$$

where $r_t = t^{1/m}$ for some $m > 0$ and $\omega \in (0, 2m)$. Then, for all $z \in \mathbb{C}_+$,

$$\left\| \chi_{B(x, r_z)} e^{-zA} \chi_{B(y, r_z)} \right\|_{p \rightarrow q} \leq |B(x, r_z)|^{1/q-1/p} R(z)^{D(1/p-1/q)} C' e^{-b'(d(x, y)/r_z)^\omega},$$

where $r_z := \cos(\arg z)^{-1/\omega} |z|^{1/m}$ and $R(z) := r_z/r_{\operatorname{Re}z} = \cos(\arg z)^{-(1/\omega+1/m)}$.

REMARK 2.2. (a) For the classical case $\omega = m/(m-1)$, $m \geq 2$, $(p, q) = (1, \infty)$ of GEs, our Theorem 2.1 corresponds to [10, Proposition 4.1].

(b) In [11], Davies verified the hypothesis of Theorem 2.1 for elliptic operators A of order $m \in 2\mathbb{N}$ on $\Omega = \mathbb{R}^D$ and for $p := (2D/(D+m)) \vee 1$, $q := p'$, $\omega = m/(m-1)$. He obtained precisely the conclusion of our Theorem 2.1; see [11, line (11) and Lemma 24].

A direct application of Theorem 2.1 is the following.

COROLLARY 2.3. *Let (Ω, μ, d) be a space of some dimension D and $p \in [1, 2)$. Let A be a non-negative selfadjoint operator on $L_2(\Omega)$ such that*

$$\left\| \chi_{B(x, r_t)} e^{-tA} \chi_{B(y, r_t)} \right\|_{p \rightarrow p'} \leq |B(x, r_t)|^{1/p'-1/p} C \exp \left(-b \left(\frac{d(x, y)}{r_t} \right)^\omega \right),$$

where $r_t = t^{1/m}$ for some $m > 1$ and $\omega \in (1, m]$. Then A has a bounded H^∞ functional calculus on $L_q(\Omega)$ for all $q \in (p, p')$.

Recall that A has a *bounded H^∞ functional calculus* if we have an estimate

$$\|f(A)\| \leq C \|f\|_{H^\infty(\Sigma_\delta)} \quad \text{for all } f \in H^\infty(\Sigma_\delta)$$

and some $\delta \in (0, \pi/2)$. Here $H^\infty(\Sigma_\delta)$ denotes the space of all bounded holomorphic functions on the sector $\Sigma_\delta := \{z; |\arg(z)| < \delta\}$. This calculus was introduced by McIntosh; its construction and applications can be found in [22, 14, 6].

PROOF OF COROLLARY 2.3. The conclusion of Theorem 2.1 holds for $q = p'$ and, obviously, A has a bounded H^∞ functional calculus on $L_2(\Omega)$. Hence the assertion follows from [6, Theorem 1.2]. \square

3. Proof of Theorem 2.1

We use the symbols \leq and \geq to indicate domination up to constants independent of the relevant parameters. Let \sim indicate the validity of \leq and \geq . While the GGEs considered in our main results enlighten the analogy to GEs, they are not convenient from a technical point of view. For this purpose, we use an equivalent type of GGE, which is provided by the following characterization [8, Proposition 2.1].

PROPOSITION 3.1. *Let (Ω, μ, d) be a space of some dimension D and $1 \leq p \leq q \leq \infty$. Let $g(s) := Ce^{-bs^\omega}$ for some $\omega > 1, b, C > 0$. Let R be a linear operator and $r > 0$.*

(i) *The following are equivalent:*

- (a) $\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{p \rightarrow q} \leq v_r(x)^{1/q-1/p} g(d(x,y)/r)$, for all $x, y \in \Omega$.
- (b) $\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{p \rightarrow v} \leq v_r(x)^{1/v-1/p} g(d(x,y)/r)$, for all $x, y \in \Omega$ and $v \in [p, q]$.
- (c) $\|\chi_{B_1} v_r^\alpha R v_r^\beta \chi_{B_2}\|_{p \rightarrow q} \leq g(d(B_1, B_2)/r)$, for all balls $B_1, B_2 \subset \Omega$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1/p - 1/q$.

(ii) *If (a) holds, then $\|v_r^\alpha R v_r^\beta\|_{u \rightarrow v} \leq C_0$, for all $p \leq u \leq v \leq q$ and all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1/u - 1/v$. Here C_0 is independent of u, v, α, β and R, r .*

The statement (i) is written modulo identification of g and \tilde{g} , where $\tilde{g}(s) := cg(s)^a$ for some constants $c, a > 0$ independent of R and r .

The preceding proposition and the following remark suggest that the right candidate for the constant $C(z)$ in complex time GGEs of the type (8) is $C(z) = R(z)^{D(1/p-1/q)}$, where $R(z) := r_z/r_{\text{Re } z}$.

REMARK 3.2. Let (Ω, μ, d) be a space of dimension $D > 0$ and $p \in [1, \infty]$, $\alpha \geq 0$. Let A be a non-negative selfadjoint operator on $L_2(\Omega)$ and $(r_z)_{z \in \mathbb{C}_+}$ a family

in \mathbb{R}_+ such that $r_z/r_{2\operatorname{Re}z} \geq \varepsilon > 0$ for all $z \in \mathbb{C}_+$. Suppose that $\|v_{r_t}^\alpha e^{-tA} v_{r_t}^\alpha\|_{p \rightarrow p'} \leq C$ for all $t \in \mathbb{R}_+$. Then $\|v_{r_z}^\alpha e^{-2zA} v_{r_z}^\alpha\|_{p \rightarrow p'} \leq C'(r_z/r_{2\operatorname{Re}z})^{2D\alpha}$ for all $z \in \mathbb{C}_+$.

PROOF. Using the norm identity $\|T\|_{p \rightarrow 2}^2 = \|T^*T\|_{p \rightarrow p'}$ in the second step, the assertion is seen as follows:

$$\begin{aligned} \|v_{r_z}^\alpha e^{-2zA} v_{r_z}^\alpha\|_{p \rightarrow p'} &\leq \|v_{r_z}^\alpha e^{-zA}\|_{2 \rightarrow p'} \|e^{-zA} v_{r_z}^\alpha\|_{p \rightarrow 2} \\ &= \|v_{r_z}^\alpha e^{-(2\operatorname{Re}z)A} v_{r_z}^\alpha\|_{p \rightarrow p'} \quad (e^{-zA} \text{ is normal, (5)}) \\ &\leq \left(\frac{r_z}{r_{2\operatorname{Re}z}}\right)^{2D\alpha} \|v_{r_{2\operatorname{Re}z}}^\alpha e^{(-2\operatorname{Re}z)A} v_{r_{2\operatorname{Re}z}}^\alpha\|_{p \rightarrow p'} \quad (\dim \Omega = D) \\ &\leq \left(\frac{r_z}{r_{2\operatorname{Re}z}}\right)^{2D\alpha} \quad (\text{by hypothesis}). \quad \square \end{aligned}$$

The last preparation for the proof of Theorem 2.1 on the extension of GGEs to complex times is the following application of the three lines lemma. The proof is a straightforward modification of [11, Lemma 9].

LEMMA 3.3. *Let $\omega, m \in \mathbb{R}_+$ such that $0 < \omega < 2m$ and $r_z := \cos(\arg z)^{-1/\omega} |z|^{1/m}$ for all $z \in \mathbb{C}_+$. Let $\delta \geq 0$ and $C_0, b > 0$. Let X be a Banach space and $F : \mathbb{C}_+ \rightarrow X$ an analytic function satisfying*

$$\begin{aligned} \|F(z)\| &\leq C_0 \cos(\arg z)^{-\delta} \quad \text{for all } z \in \mathbb{C}_+, \\ \|F(t)\| &\leq C_0 \exp(-br_t^{-\omega}) \quad \text{for all } t \in \mathbb{R}_+. \end{aligned}$$

Then we have $\|F(z)\| \leq C_0 2^\delta \cos(\arg z)^{-\delta} e^{-(\omega/2m)br_z^{-\omega}}$ for all $z \in \mathbb{C}_+$.

PROOF. We define for all $\gamma \in (0, \pi/2)$:

$$\begin{aligned} g_\gamma^\pm(z) &:= C_0 \exp\left(-b \frac{e^{\mp i(\pi/2 - \omega\gamma/m)}}{\sin(\omega\gamma/m)} z^{-\omega/m}\right), \quad \pm \arg z \in [0, \gamma], \\ F_\gamma^\pm(z) &:= F(z) g_\gamma^\pm(z)^{-1}, \quad \pm \arg z \in [0, \gamma]. \end{aligned}$$

By hypothesis, we have for all $\gamma \in (0, \pi/2)$ and $t > 0$:

$$\begin{aligned} \|F_\gamma^\pm(t)\| &= \|F(t)\| C_0^{-1} \exp(bt^{-\omega/m}) \leq 1, \\ \|F_\gamma^\pm(e^{\pm i\gamma} t)\| &= \|F(e^{\pm i\gamma} t)\| C_0^{-1} \leq \cos(\gamma)^{-\delta}. \end{aligned}$$

Hence the three lines lemma yields, for all $\gamma \in (0, \pi/2)$ and $\theta \in [0, 1]$,

$$\|F_\gamma^\pm(e^{\pm i\gamma\theta} t)\| \leq 1^{1-\theta} \cos(\gamma)^{-\delta\theta} \leq \cos(\gamma)^{-\delta}.$$

In other words, we have for all $\phi \in [0, \pi/2)$, $t > 0$ and $\gamma \in [\phi, \pi/2)$,

$$\begin{aligned} \|F(e^{\pm i\phi}t)\| &= \|F_\gamma^\pm(e^{\pm i\phi}t)\| \|g_\gamma^\pm(e^{\pm i\phi}t)\| \\ &\leq \cos(\gamma)^{-\delta} C_0 \exp\left(-b \sin\left(\frac{\omega}{m}(\gamma - \phi)\right) t^{-\omega/m}\right). \end{aligned}$$

It remains to choose $\gamma = \pi/4 + \phi/2$ and to deduce from the concavity of \sin on $[0, \pi/2]$:

$$\begin{aligned} \cos(\gamma) &= \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \geq \frac{1}{2} \sin\left(\frac{\pi}{2} - \phi\right) = \frac{1}{2} \cos(\phi), \\ \sin\left(\frac{\omega}{m}\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right) &\geq \frac{\omega}{2m} \sin\left(\frac{\pi}{2} - \phi\right) = \frac{\omega}{2m} \cos(\phi) = \frac{\omega}{2m} r_{e^{\pm i\phi}t}^{-\omega} t^{\omega/m}. \quad \square \end{aligned}$$

PROOF OF THEOREM 2.1. We recall the basic volume estimate

$$(9) \quad v_s(x) \leq C(1 + s/r)^D v_r(x) \quad \text{for all } x \in \Omega, \quad r, s > 0.$$

Now fix $\alpha := 1/p_o - 1/q$ and $\beta := 1/p - 1/p_o$. By Proposition 3.1 (ii), we obtain

$$\|v_{r_t}^\alpha e^{-tA}\|_{p_o \rightarrow q}, \|e^{-tA} v_{r_t}^\beta\|_{p \rightarrow p_o} \leq C \quad \text{for all } t > 0.$$

This allows for all $z \in \mathbb{C}_+$, $t \in (0, \operatorname{Re} z/2)$ and $s > 0$ the estimate

$$\begin{aligned} \|v_s^\alpha e^{-zA} v_s^\beta\|_{p \rightarrow q} &\leq \|v_s^\alpha e^{-tA}\|_{p_o \rightarrow q} \|e^{-(z-2t)A}\|_{p_o \rightarrow p_o} \|e^{-tA} v_s^\beta\|_{p \rightarrow p_o} \\ &\leq \left(1 + \frac{s}{r_t}\right)^{D\alpha} \|v_{r_t}^\alpha e^{-tA}\|_{p_o \rightarrow q} \left(1 + \frac{s}{r_t}\right)^{D\beta} \|e^{-tA} v_{r_t}^\beta\|_{p \rightarrow p_o} \quad (\text{by (9)}) \\ &\leq \left(1 + \frac{s}{r_t}\right)^{D(1/p-1/q)}, \quad \text{since } \alpha + \beta = 1/p - 1/q. \end{aligned}$$

Denoting $z = e^{i\theta}r$ for $z \in \mathbb{C}_+$, we note that

$$\begin{aligned} 1 + \frac{s}{(\operatorname{Re} z/2)^{1/m}} &\leq (2/\cos\theta)^{1/m} \left(1 + \frac{s}{r^{1/m}}\right) \\ &\leq (2/\cos\theta)^{1/m} \sqrt{2} \left|1 + \frac{s}{z^{1/m}}\right| \\ &= C \cos(\arg z)^{-1/m} \left|1 + \frac{s}{z^{1/m}}\right| \quad \text{for all } z \in \mathbb{C}_+, \quad s > 0, \end{aligned}$$

which combines with the preceding estimate for $t \nearrow \operatorname{Re} z/2$ to give

$$\|v_s^\alpha e^{-zA} v_s^\beta\|_{p \rightarrow q} \leq \cos(\arg z)^{-(D/m)(1/p-1/q)} \left|1 + \frac{s}{z^{1/m}}\right|^{D(1/p-1/q)}.$$

Denote $g(u) := Ce^{-bu^\omega}$ and fix balls $B_1, B_2 \subset \Omega$. We have for all $s, t > 0$,

$$\begin{aligned} \|\chi_{B_1} v_s^\alpha e^{-tA} v_s^\beta \chi_{B_2}\|_{p \rightarrow q} &\leq \left(1 + \frac{s}{r_t}\right)^{D(\alpha+\beta)} \|\chi_{B_1} v_{r_t}^\alpha e^{-tA} v_{r_t}^\beta \chi_{B_2}\|_{p \rightarrow q} \quad (\text{by (9)}) \\ &\leq \left(1 + \frac{s}{r_t}\right)^{D(1/p-1/q)} g(d(B_1, B_2)r_t^{-1}), \end{aligned}$$

since $\alpha + \beta = 1/p - 1/q$. Here we used the hypothesis, and in the last step, Proposition 3.1 (i). For fixed $s > 0$, we consider the analytic function $F : \mathbb{C}_+ \rightarrow \mathcal{L}(L_p(\Omega), L_q(\Omega))$ defined by

$$F(z) := \left(1 + \frac{s}{z^{1/m}}\right)^{-D(1/p-1/q)} \chi_{B_1} v_s^\alpha e^{-zA} v_s^\beta \chi_{B_2}.$$

We have shown the following two bounds:

$$\begin{aligned} \|F(t)\|_{p \rightarrow q} &\leq g(d(B_1, B_2)t^{-1/m}) \quad \text{for all } t > 0, \\ \|F(z)\|_{p \rightarrow q} &\leq \cos(\arg z)^{-(D/m)(1/p-1/q)} \quad \text{for all } z \in \mathbb{C}_+. \end{aligned}$$

We apply Lemma 3.3 (to $g(d(B_1, B_2)\cdot)$ instead of g) and obtain

$$\|F(z)\|_{p \rightarrow q} \leq \cos(\arg z)^{-\frac{D}{m}(1/p-1/q)} g(d(B_1, B_2)r_z^{-1}) \quad \text{for all } z \in \mathbb{C}_+,$$

where $r_z = \cos(\arg z)^{-1/\omega} |z|^{1/m}$. This means for all $z \in \mathbb{C}_+$ and $s > 0$,

$$\begin{aligned} &\|\chi_{B_1} v_s^\alpha e^{-zA} v_s^\beta \chi_{B_2}\|_{p \rightarrow q} \\ &\leq \left|1 + \frac{s}{z^{1/m}}\right|^{D(1/p-1/q)} \cos(\arg z)^{-(D/m)(1/p-1/q)} g(d(B_1, B_2)r_z^{-1}). \end{aligned}$$

Choosing $s = r_z$ yields

$$\begin{aligned} &\|\chi_{B_1} v_{r_z}^\alpha e^{-zA} v_{r_z}^\beta \chi_{B_2}\|_{p \rightarrow q} \\ &\leq \left|1 + \frac{r_z}{z^{1/m}}\right|^{D(1/p-1/q)} \cos(\arg z)^{-(D/m)(1/p-1/q)} g(d(B_1, B_2)r_z^{-1}) \\ &\leq \cos(\arg z)^{-D(1/\omega+1/m)(1/p-1/q)} g(d(B_1, B_2)r_z^{-1}). \end{aligned}$$

By Proposition 3.1 (i) and $\alpha + \beta = 1/p - 1/q$, the above estimate is equivalent to

$$\begin{aligned} &\|\chi_{B(x, r_z)} e^{-zA} \chi_{B(y, r_z)}\|_{p \rightarrow q} \\ &\leq v_{r_z}(x)^{1/q-1/p} \cos(\arg z)^{-D(1/\omega+1/m)(1/p-1/q)} g\left(\frac{d(x, y)}{r_z}\right). \end{aligned} \quad \square$$

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