THE WEIGHTED $g$-DRAZIN INVERSE FOR OPERATORS

A. DAJIĆ and J. J. KOLIHA

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Abstract

The paper introduces and studies the weighted $g$-Drazin inverse for bounded linear operators between Banach spaces, extending the concept of the weighted Drazin inverse of Rakočević and Wei (Linear Algebra Appl. 350 (2002), 25–39) and of Cline and Greville (Linear Algebra Appl. 29 (1980), 53–62). We use the Mbekhta decomposition to study the structure of an operator possessing the weighted $g$-Drazin inverse, give an operator matrix representation for the inverse, and study its continuity. An open problem of Rakočević and Wei is solved.


Keywords and phrases: Banach algebra without unit, $g$-Drazin inverse, bounded linear operator, weighted $g$-Drazin inverse, the Mbekhta decomposition, ascent and descent.

1. Introduction

In recent papers [13, 14], Rakočević and Wei defined and investigated the weighted Drazin inverse for bounded linear operators between Banach and Hilbert spaces, extending the concept of a weighted Drazin inverse for rectangular matrices introduced by Cline and Greville [5]. The weighted Drazin inverse for operators was previously introduced and studied by Qiao in [12], and further investigated by Wang in [16, 17].

The main purpose of this paper is to introduce and study the weighted $g$-Drazin inverse for bounded linear operators between Banach spaces $X$ and $Y$, thus further extending the above mentioned works.

Let $\mathcal{B}(X, Y)$ denote the set of all bounded linear operators between $X$ and $Y$, and let $W$ be a nonzero operator in $\mathcal{B}(Y, X)$. The $W$-weighted $g$-Drazin inverse (the $Wg$-Drazin inverse for short) can be studied in the framework of Banach algebras when we introduce on the space $\mathcal{B}(X, Y)$ the $W$-product $A \star B = AWB$, and the
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\begin{align*}
W\text{-norm } \| A \|_W = \| A \| \| W \|. \quad & \text{This elegant approach which turns } \mathcal{B}(X, Y) \text{ into a Banach algebra was suggested to the authors of [13, 14] by an anonymous referee. Unless } W \text{ is invertible (and this would require the spaces } X \text{ and } Y \text{ to be isomorphic and homeomorphic), the resulting algebra is without unit.}

\text{In our work we remove the restriction of finite polarity of the operator } WA \text{ (and } AW) \text{ adopted by Rakočević and Wei [13]. In addition, we solve an open problem posed in [13], and complete and extend the results of Buoni and Faires [3] on the ascent and descent of } AB \text{ and } BA.

\text{In Section 2 we gather relevant results on the } g\text{-Drazin inverse in Banach algebras without unit in order to study the } Wg\text{-Drazin inverse within the space } \mathcal{B}(X, Y), \text{ without having to adjoin a unit. Section 3 introduces and studies the weighted } g\text{-Drazin inverse between two different Banach spaces. In Section 4 we explore some properties of the weighted } g\text{-Drazin inverse, including the core decomposition and an integral representation for the weighted inverse. The ascent and descent for } WA \text{ and } AW \text{ is studied in Section 5, and a solution to an open problem posed by Rakočević and Wei in [13] is given there. In Section 6 we compare the Mbekhta decomposition for the operators } WA \text{ and } AW \text{ and recover and sharpen a result of Yukhno [19] on rectangular matrices. In the remaining sections we give an operator matrix representation for the } Wg\text{-Drazin inverse, compare it with the Moore–Penrose inverse in Hilbert spaces, and give necessary and sufficient conditions for its continuity.}

2. The } g\text{-Drazin inverse in Banach algebras without unit}

\text{Let } \mathcal{A} \text{ be a Banach algebra. We write } \mathcal{A}^{qnil} \text{ for the set of all quasinilpotent elements in } \mathcal{A}, \text{ that is, elements } a \text{ satisfying } \lim_{n \to \infty} \| a^n \|^{1/n} = 0; \text{ the set of all nilpotent elements is denoted by } \mathcal{A}^{nil}. \text{ If } \mathcal{A} \text{ is unital, we denote by } \mathcal{A}^{inv} \text{ the group of all invertible elements in } \mathcal{A}. \text{ An element } a \in \mathcal{A} \text{ is quasipolar if } 0 \text{ is not an accumulation point of the spectrum of } a. \text{ In an algebra without unit, this is equivalent to } 0 \text{ being an isolated spectral point of } a. \text{ The set of all quasipolar elements of } \mathcal{A} \text{ will be denoted by } \mathcal{A}^{qpol}. \text{ An element } a \in \mathcal{A} \text{ is polar if it is quasipolar and } 0 \text{ is at most a pole of the resolvent of } a. \text{ The set of all polar elements is denoted by } \mathcal{A}^{pol}.

\text{The following holds [7, Theorems 4.2 and 5.1]:}

**Lemma 2.1.** Let } \mathcal{A} \text{ be a unital Banach algebra. Then } a \in \mathcal{A} \text{ is quasipolar (polar) in } \mathcal{A} \text{ if and only there exists } p \in \mathcal{A} \text{ such that}

\begin{equation}
(2.1) \quad p^2 = p, \quad ap = pa \in \mathcal{A}^{qnil} \quad (ap = pa \in \mathcal{A}^{nil}), \quad a + p \in \mathcal{A}^{inv}.
\end{equation}

\text{The resolvent } R(\lambda; a) = (\lambda I - a)^{-1} \text{ has a Laurent expansion in some punctured}
neighbourhood $0 < |\lambda| < r$ of 0 given by

$$R(\lambda; a) = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n p - \sum_{n=1}^{\infty} \lambda^{n-1} b^n,$$

where $b = (a + p)^{-1}(1 - p)$.

The element $p$ is uniquely determined by the conditions of the theorem; it is called the \textit{spectral idempotent} of $a$, and it double commutes with $a$. The element $q = 1 - p$ is the \textit{support idempotent} of $a$. The support idempotent of a quasipolar element exists in an algebra without a unit, but not the spectral idempotent. The element $b = (a + p)^{-1}(1 - p)$ defines the \textit{g-Drazin inverse} of $a$ in the case of a unital algebra; $b$ also double commutes with $a$. We write $a^s$ and $a^w$ for the spectral idempotent and the support idempotent of a quasipolar element $a$, respectively.

From now on we assume that $\mathcal{A}$ is a complex Banach algebra without unit.

The \textit{unitisation} of $\mathcal{A}$ is the unital Banach algebra $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$ containing $\mathcal{A}$ as a two sided ideal of codimension 1 [2, page 15]. Given $a \in \mathcal{A}$, we define the \textit{spectrum} $\text{Sp}(a)$ of $a$ in $\mathcal{A}$ as the spectrum of $a$ considered as an element of the unital Banach algebra $\mathcal{A}_1$, that is, the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - a \notin \mathcal{A}_1^{\text{inv}}$. Observe that 0 is always in the spectrum of any element of a Banach algebra without unit.

\textbf{Proposition 2.2.} Let $\mathcal{A}$ be a Banach algebra without unit. Then $a \in \mathcal{A}^{\text{qpol}}$ (a $\in \mathcal{A}^{\text{pol}}$) if and only if there exists $b \in \mathcal{A}$ such that

$$ab = ba, \quad bab = b, \quad a - aba \in \mathcal{A}^{\text{nil}} (a - aba \in \mathcal{A}^{\text{qnil}}).$$

The element $b$, if it exists, is unique.

\textbf{Proof.} We embed $\mathcal{A}$ into its unitisation $\mathcal{A}_1$.

If $a$ is quasipolar in $\mathcal{A}$, then it is also quasipolar in $\mathcal{A}_1$. Let $p$ be the spectral idempotent of $a$ in $\mathcal{A}_1$, and $b = (a + p)^{-1}(1 - p)$ the Drazin inverse of $a$ in $\mathcal{A}_1$. Since $1 - p$ is in $\mathcal{A}$, so is $b$ ($\mathcal{A}$ is an ideal). The equations (2.3) are then easily verified.

Conversely, let equations (2.3) hold. Then $p = 1 - ab$ is the spectral idempotent of $a$ in $\mathcal{A}_1$ [7, Theorem 4.2], and $a$ is quasipolar, both in $\mathcal{A}_1$ and $\mathcal{A}$. From

$$(a + p)b = (a + 1 - ab)b = ab + b - bab = ab = 1 - p$$

and the invertibility of $a + p$ in $\mathcal{A}_1$, we get $b = (a + p)^{-1}(1 - p)$ in $\mathcal{A}_1$ (and in $\mathcal{A}$). This proves the uniqueness of $b$ satisfying (2.3).

\textbf{Definition 2.3.} Let $\mathcal{A}$ be a Banach algebra without unit and let $a \in \mathcal{A}^{\text{qpol}}$. We define the \textit{g-Drazin inverse} $a^D$ of $a$ to be the unique element $b$ satisfying (2.3). The \textit{Drazin index} of a quasipolar element $a$ is defined by

$$i(a) = \inf \{ k \in \mathbb{N} : (a - a^2 a^D)^k = 0 \}$$
(inf $\emptyset = \infty$). The $g$-Drazin inverse of a polar element is called the Drazin inverse.

We observe that $a \in \mathcal{A}$ is polar if and only if it is quasipolar and has a finite Drazin index.

As in the unital case, any $g$-Drazin invertible element $a$ of $\mathcal{A}$ has the ‘core’ decomposition.

**Proposition 2.4.** Let $\mathcal{A}$ be a Banach algebra without unit. Then $a \in \mathcal{A}^{qpol}$ if and only if $a = c + u$, where $c$ is simply polar, $u$ quasinilpotent, and $cu = 0 = uc$. Such a decomposition is unique. In addition,

\begin{equation}
(2.4)
    a^D = c^D, \quad a^\sigma = c^\sigma, \quad \text{Sp}(c) = \text{Sp}(a).
\end{equation}

We can show that $ua^\sigma = 0$ and that the element $c$, called the core of $a$, satisfies

\[ c = a a^\sigma = (a^D)^D = a^2 a^D. \]

**Proposition 2.5.** Let $\mathcal{A}$ be a Banach algebra without unit and let $a \in \mathcal{A}^{qpol}$. Then $a^D = a$ if and only if $a^3 = a$.

**Proof.** Suppose that $a^3 = a$ and let $a = c + u$ be the core decomposition of $a$. We observe that $a^3 = c^3 + u^3$ is the core decomposition for $a^3 = a$. From the uniqueness, $c^3 = c$ and $u^3 = u$. Since $u^3 = u \in \mathcal{A}^{qnil}$, we conclude that $u = 0$:

\[
\lim_{n \to \infty} \|u\|^{1/3^n} = \lim_{n \to \infty} \|u^{3^n}\|^{1/3^n} = r(u) = 0.
\]

Thus $a = c = a a^\sigma$ is simply polar, and

\[ a^D = (a^D)^2 a = (a^D)^2 a^3 = (a^D a^2)(a^D a) = a a^\sigma = a. \]

Conversely, if $a^D = a$, then $a = (a^D)^2 a = a^3$. \hfill \square

As an example of further properties of the $g$-Drazin inverse in Banach algebras without unit we prove the following result, which for matrices reduces to Theorem 7.8.4 of Campbell and Meyer [4].

**Proposition 2.6.** Let $\mathcal{A}$ be a Banach algebra without unit, and let $a, b \in \mathcal{A}$ be such that $(ba)^2 \in \mathcal{A}^{qpol}$. Then both $ab$ and $ba$ are $g$-Drazin invertible, and

\begin{equation}
(2.5)
    (ab)^D = a((ba)^D)b.
\end{equation}
3. The weighted g-Drazin inverse for operators

Throughout this section we assume that $X, Y$ are nonzero complex Banach spaces and $W$ is a fixed nonzero operator in $B(Y, X)$, the set of all bounded linear operators on $Y$ to $X$. First we turn $B(X, Y)$ into a Banach algebra $B_w(X, Y)$ (in general without a unit) by introducing a multiplication of elements of $B(X, Y)$ facilitated by the operator $W$, and imposing a suitable norm on $B(X, Y)$.

**Lemma 3.1.** Let $B_w(X, Y)$ be the space $B(X, Y)$ equipped with the multiplication

$$A \star B = AWB,$$

and norm $\|A\|_w = \|A\|\|W\|$. Then $B_w(X, Y)$ becomes a complex Banach algebra; $B_w(X, Y)$ has a unit if and only if $W$ is invertible, in which case $W^{-1}$ is that unit.

**Proof.** The verification of most Banach algebra axioms is straightforward. The positive definiteness of the norm is ensured by the fact that $W \neq 0$. We check the submultiplicativity of the norm. If $A, B \in B(X, Y)$, then

$$\|A \star B\|_w = \|AWB\| \leq \|A\|\|W\|\|B\| = \|A\|_w\|B\|_w.$$

If $W$ is invertible, then $W^{-1} \in B(X, Y)$ is the unit in $B_w(X, Y)$. Conversely, assume that $P \in B(X, Y)$ is the unit for $B_w(X, Y)$. Then

$$AWP = A = PWA \quad \text{for all } A \in B(X, Y).$$

For each $y \in Y$ and $f \in X^*$ define $f \otimes y : X \to Y$ by $(f \otimes y)x = f(x)y$ for all $x \in X$; then $f \otimes y \in B(X, Y)$. From $PW(f \otimes y)x = (f \otimes y)x$ we get $f(x)PWy = f(x)y$. Selecting $x$ and $f$ so that $f(x) \neq 0$, we obtain $PWy = y$ for any $y \in Y$. From $(f \otimes y)WPx = (f \otimes y)x$ we get $f(WPx)y = f(x)y$. Selecting $y \neq 0$, yields $f(WPx) = f(x)$ for all $f \in X^*$, which implies $WPx = x$ for any $x \in X$. Then $W$ is invertible; setting $A = W^{-1}$ in (3.3), we get $W^{-1} = PWW^{-1} = P$. 

PROOF. If $(ba)^2 \in A_{qnil}$, then also $(ab)^2$, $ab$ and $ba$ are quasipolar, and $w = ((ba)^2)^D = (ba)^D$ commutes with $ba$. Set $c = a((ba)^2)^D = awb$. It is not difficult to show that $(ab)c = c(ab)$ and $(ab)c^2 = c$. The element $ab - (ab)^2c = (a - a(ba)^2w)b$ is quasinilpotent if and only if $x = b(a - a(ba)^2w) = ba - (ba)^2w$ is quasinilpotent. Imbedding $A$ into its unitisation $A_1$, we recall that $p = 1 - (ba)^2w$ is idempotent; hence $x = (ba)p \in A_{qnil}$ if and only if $x^2 = (ba)^2p \in A_{qnil}$ if and only if $(ba)^2 - (ba)^4w \in A_{qnil}$. This completes the proof.
We observe that if $\mathcal{B}_w(X, Y)$ has the unit $W^{-1}$, the spaces $X$ and $Y$ are isomorphic and homeomorphic; in particular, $X$ and $Y$ are of the same dimension. Moreover, the norm of the unit in $\mathcal{B}_w(X, Y)$ is equal to $\|W^{-1}\|_w = \|W^{-1}\|W = \kappa(W)$, known as the condition number of $W$.

For any $n \in \mathbb{N}$ we write $A^n = A \ast \cdots \ast A$ ($n$ factors). Observe that

$$\tag{3.4} A^n = (AW)^n A = A(WA)^{n-1}.$$ 

We write $r_w(\cdot)$ for the spectral radius of elements of $\mathcal{B}_w(X, Y)$. We show that

$$\tag{3.5} r_w(A) = r(AW) = r(WA),$$

where $r(\cdot)$ is the spectral radius in $\mathcal{B}(Y)$ or $\mathcal{B}(X)$. Indeed,

$$r(AW) = \lim_{n \to \infty} \|(AW)^n\|^{1/n} \leq \lim_{n \to \infty} \|((AW)^n)^{1/n} A\|W\|^{1/n} = \lim_{n \to \infty} \|A^n\|^{1/n} = r_w(A).$$

Conversely,

$$r_w(A) = \lim_{n \to \infty} \|A^n\|^{1/n} = \lim_{n \to \infty} \|A^n\|^{1/n} W\|^{1/n} = \lim_{n \to \infty} \|(AW)^n A\|^{1/n} W\|^{1/n} \leq \lim_{n \to \infty} \|(AW)^{n-1}\|^{1/n} \lim_{n \to \infty} (\|A\|W\|)^{1/n} = \lim_{n \to \infty} \|(AW)^{n-1}\|^{1/n} = r(AW)$$

as $\lim_{n \to \infty} \|(AW)^{n-1}\|^{1/n} = \lim_{n \to \infty} \|(AW)^n\|^{1/n}$. The second equality in (3.5) follows by symmetry.

**Definition 3.2.** Let $W$ be a fixed nonzero operator in $\mathcal{B}(Y, X)$. An operator $A \in \mathcal{B}(X, Y)$ is called Wg-Drazin invertible if $A$ is quasipolar in the Banach algebra $\mathcal{B}_w(X, Y)$. The Wg-Drazin inverse $A^{D,w}$ of $A$ (or W-weighted g-Drazin inverse) is then defined as the g-Drazin inverse $B$ of $A$ in the Banach algebra $\mathcal{B}_w(X, Y)$; $i_w(A)$ is the Drazin index of $A$ in $\mathcal{B}_w(X, Y)$. A polar element of $\mathcal{B}_w(X, Y)$ is called W-Drazin invertible, with the W-Drazin inverse $A^{D,w} = B$.

The Wg-Drazin inverse is unique if it exists (Proposition 2.2), and is characterised by the following theorem.

**Theorem 3.3.** Let $W$ be a fixed nonzero operator in $\mathcal{B}(Y, X)$. Then $A \in \mathcal{B}(X, Y)$ is Wg-Drazin invertible with the Wg-Drazin inverse $A^{D,w} = B \in \mathcal{B}(X, Y)$ if and only if one of the following equivalent conditions holds:

(i) $AW$ is quasipolar in $\mathcal{B}(Y)$ with $(AW)^D = BW$;
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(ii) \( WA \) is quasipolar in \( \mathcal{B}(X, Y) \) with \( (WA)^D = WB \);

(iii) There exists \( B \in \mathcal{B}(X, Y) \) satisfying

\[
(AW)B = (BW)A, \quad (BW)^2 A = B, \quad (AW)^2 BW - AW \in \mathcal{B}(Y)^{qnil};
\]

(iv) There exists \( B \in \mathcal{B}(X, Y) \) satisfying

\[
A(WB) = B(WA), \quad A(WB)^2 = B, \quad WB(WA)^2 - WA \in \mathcal{B}(X)^{qnil}.
\]

The \( Wg \)-Drazin inverse \( A^{D,W} \) of \( A \) then satisfies

\[
(3.6) \quad A^{D,W} = ((AW)^D)^2 A = A((WA)^D)^2.
\]

**Proof.** Suppose that \( A \) has the \( Wg \)-Drazin inverse \( B \).

The conditions

\[
A \bullet B = B \bullet A, \quad B \bullet A \bullet B = B, \quad A \bullet B \bullet A - A \in \mathcal{B}_w(X, Y)^{qnil},
\]

translate to

\[
(3.7) \quad AWB = BWA, \quad (BW)^2 A = B, \quad T = (AW)^2 B - A \in \mathcal{B}_w(X, Y)^{qnil}.
\]

Let \( C = BW \). Then \( (AW)C = C(AW) \) and \( C^2(AW) = C \) by (3.7). Finally, by (3.5), \( r(TW) = r_w(T) = 0 \). Hence \( (AW)^2 C - AW = TW \) is quasinilpotent in \( \mathcal{B}(Y) \), and (i) is proved.

Condition (ii) follows from a symmetrical argument. Conditions (i) and (iii) (respectively (ii) and (iv)) are equivalent by the characterisation of the \( g \)-Drazin inverse given in Proposition 2.2.

Conversely, suppose that \( AW \in \mathcal{B}(Y) \) has the \( g \)-Drazin inverse \( C \). Let \( B = C^2 A \). The equations \( (AW)C = C(AW) \) and \( C^2(AW) = C \) imply

\[
A \bullet B = AWC^2 A = C^2 AWA = B \bullet A, \quad \text{and}
\]

\[
B \bullet A \bullet B = (C^2AW)(AWC^2)A = C^2 A = B.
\]

Write \( A \bullet B \bullet A - A = (AWC^2)AWA - A = CAWA - A = S \). Since \( SW = C(AW)^2 - AW \) is quasinilpotent in \( \mathcal{B}(Y) \), \( r_w(S) = r(SW) = 0 \), and \( S \) is quasinilpotent in \( \mathcal{B}_w(X, Y) \). This proves that condition (i) implies that \( A \) is \( Wg \)-Drazin invertible with \( A^{D,W} = C^2 A \). The rest follows from Proposition 2.2 by symmetry.

From (3.6) we find an expression for the support idempotent \( A^{\circ,W} \) of \( A \) in \( \mathcal{B}_w(X, Y) \):

\[
A^{\circ,W} = A \bullet A^{D,W} = AW((AW)^D)^2 A = (AW)^D A.
\]

By symmetry,

\[
(3.8) \quad A^{\circ,W} = (AW)^D A = A(WA)^D.
\]
PROPOSITION 3.4. If \( A \in \mathcal{B}(X, Y) \) is \( W_{g} \)-Drazin invertible, then the Drazin indices \( i_{W}(A) \), \( i(WA) \), and \( i(AW) \) are all finite or all infinite, and satisfy the inequalities

\[
\max \{i(AW), i(WA)\} \leq i_{W}(A) \leq \min \{i(AW), i(WA)\} + 1.
\]

PROOF. Let \( A^{D,W} = B \) be the \( W_{g} \)-Drazin inverse of \( A \) and let \( T = (AW)^{2}B - A \). If \( i_{W}(A) = k < \infty \), then \( T^{*} = 0 \). Consequently \((TW)^{k} = (TW)^{k-1}TW = T^{*}W = 0\) and hence \( i(AW) \leq i_{W}(A) \).

Let \( AW \) have the \( g \)-Drazin inverse \( C \) and let \( S = CAWA - A \). If \( i(AW) = k < \infty \), then \((SW)^{k} = 0\), and \( S^{*}(k+1) = (SW)^{k}S = 0\), that is, \( i_{W}(A) \leq k + 1 \). This proves the inequality for \( i(AW) \) in (3.9).

It is known that for any \( A \in \mathcal{B}(X, Y) \) and \( W \in \mathcal{B}(Y, X) \),

\[
\text{Sp}(AW) \setminus \{0\} = \text{Sp}(WA) \setminus \{0\}.
\]

Hence \( AW \) is \( g \)-Drazin invertible in \( \mathcal{B}(Y) \) if and only if \( WA \) is \( g \)-Drazin invertible in \( \mathcal{B}(X) \). The inequality for \( i(WA) \) in (3.9) is obtained by symmetry.

EXAMPLE 3.5. The inequality \( i(AW) \leq i_{W}(A) \) (respectively \( i(WA) \leq i_{W}(A) \)) in (3.9) can be strict. Let

\[
W = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix},
\]

and let \( B_{w} \) be the space \( \mathcal{M}_{2,3}(\mathbb{C}) \) of all complex \( 2 \times 3 \) matrices with the multiplication (3.1). By the preceding theorem, every element \( A \in \mathcal{M}_{2,3}(\mathbb{C}) \) has a \( g \)-Drazin inverse of finite Drazin index in \( \mathcal{B}_{w} \) since the matrix \( AW \) has the conventional Drazin inverse \((AW)^{D} \) in \( \mathcal{M}_{2,3}(\mathbb{C}) \). Let

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Then \( AW = 0 = B \), where \( B \) is the \( g \)-Drazin inverse of \( A \) in \( \mathcal{B}_{w} \), and \( T = (AW)^{2}B - A = -A \). Since \( T \neq 0 \) and \( T \cdot T = AWA = 0 \), we have \( i_{W}(A) = 2 \). On the other hand, \( i(AW) = i(0) = 1 \). An example of a strict inequality between \( i(WA) \) and \( i_{W}(A) \) can be obtained from the present example and the following proposition involving the dual spaces \( X^{*} \) and \( Y^{*} \) of \( X \) and \( Y \).

PROPOSITION 3.6. \( A \in \mathcal{B}(X, Y) \) is \( W_{g} \)-Drazin invertible if and only if the adjoint \( A^{*} \in \mathcal{B}(Y^{*}, X^{*}) \) of \( A \) is \( W^{*g} \)-Drazin invertible. In this case

\[
(A^{*})^{D,W^{*}} = (A^{D,W})^{*}.
\]
**Proof.** Since $\text{Sp}(AW)^* = \text{Sp}(AW)$, $(AW)^*$ is quasipolar if and only if $AW$ is quasipolar. Hence $W^*A^* = (AW)^*$ is $g$-Drazin invertible if and only if $AW$ is. By Theorem 3.3, $A^*$ is $W^*g$-Drazin invertible if and only if $A$ is $Wg$-Drazin invertible. Equation (3.11) follows on application of Proposition 2.2.

**Example 3.7 (Rakočević and Wei [13]).** If $A \in \mathcal{B}(X, Y)$ is a finite rank operator, then $A$ has a finite index $Wg$-Drazin inverse for any nonzero $W \in \mathcal{B}(Y, X)$. If $W \in \mathcal{B}(Y, X)$ is a nonzero operator of finite rank, then any $A \in \mathcal{B}(X, Y)$ has a finite index $Wg$-Drazin inverse.

### 4. Further properties of the $Wg$-Drazin inverse

First we briefly explore a duality between $A^{D, W}$ and $W^{D, A}$ provided $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)$. From Theorem 3.3 we see that the weighted $g$-Drazin inverse $W^{D, A}$ exists if and only if $A^{D, W}$ exists. Equation (3.6) gives rise to the following relations:

$$W^{D, A} = (WA)^D = WA^{D, W},$$
$$AW^{D, A} = (AW)^D = A^{D, W}W.$$

We can then express $W^{D, A}$ in terms of $A^{D, W}$ and vice versa:

$$W^{D, A} = WA^{D, W}WA^{D, W}W,$$
$$A^{D, W} = AW^{D, A}AW^{D, A}.$$

(4.1)

We observe that in (4.1), the operators $AW^{D, A}$ and $W^{D, A}$ are simply polar (that is, of index 1 or 0): for example, $AW^{D, A} = AW((AW)^D)^2 = (AW)^D$. The simple polarity of the $g$-Drazin inverse of $AW$ is well known (see [7]). Specialised to matrices, this proves the necessary part of Theorem 3 in [5].

**Proposition 4.1.** Let $A \in \mathcal{B}(X, Y)$ be $Wg$-Drazin invertible. Then the following are true:

(i) $A = A^{D, W}$ if and only if $A = A^{\ast 3} = AWAWA$.
(ii) $(A^{D, W})^{D, W} = (AW)^{\gamma}A = A(WA)^{\gamma}$.
(iii) $(A^{D, W})^{\circ, W} = A^{\circ, W}$.
(iv) For any $n \in \mathbb{N}$, $(A^{D, W})^{M} = ((AW)^{D})^{n+1}A = A((WA)^{D})^{n+1} = (A^{M})^{D, W}$.

**Proof.** (i) This follows from Proposition 2.5 applied to $\mathcal{B}_W(X, Y)$.
(ii) Applying the results of [7] while working in the Banach algebra $\mathcal{B}_W(X, Y)$, we have $(A^{D, W})^{D, W} = A^{\ast}A^{\ast W} = AW(AW)^{D, W}A = (AW)^{\gamma}A$.
(iii) In the proof of [7, Theorem 5.2] it is shown that a quasipolar element and its $g$-Drazin inverse have the same support idempotent.
(iv) This is shown via induction on $n$. \qed
Part (ii) of the preceding theorem implies that \((A^{D,W})^{D,W} = A\) if and only if 
\((AW)^\circ A = A (AWA)^\circ = A\). This is equivalent to \(A\) being simply polar in 
\(B_w(X, Y)\).

From [7, Theorem 5.5] we can deduce the following result.

**Proposition 4.2.** Let \(A, B \in \mathcal{B}(X, Y)\) be \(Wg\)-Drazin invertible. If \(AWB = BW A\), 
then \(AWB\) is \(Wg\)-Drazin invertible with \((AWB)^{D,W} = A^{D,W}WB^{D,W}\).

We now turn our attention to an analogue of the core decomposition for the weighted 
g-Drazin inverse.

**Theorem 4.3.** An operator \(A \in \mathcal{B}(X, Y)\) is \(Wg\)-Drazin invertible if and only if 
there exist operators \(C, U \in \mathcal{B}(X, Y)\) such that

\[
(A) \quad A = C + U, \quad CWU = 0, \quad UWC = 0, \quad (CW)^\circ C = C, \quad UW \in \mathcal{B}(Y)^{qnil}.
\]

Such operators are uniquely determined, and \(C = (A^{D,W})^{D,W} = (AW)^\circ A\). Further,

\[
(B) \quad (AW)^D = (CW)^D, \quad (AW)^\circ = (CW)^\circ, \quad \text{Sp}(AW) \cup \{0\} = \text{Sp}(CW).
\]

**Proof.** We apply Theorem 2.4 to \(B_w(X, Y)\). \(A\) is \(Wg\)-Drazin invertible if and only if 
there exist \(C, U \in \mathcal{B}(X, Y)\) such that \(A = C + U, \quad C \star U = CWU = 0, \quad U \star C = UWC = 0\), \(C\) is 
simply polar in \(B_w(X, Y)\), and \(U\) is quasinilpotent in \(B_w(X, Y)\). The element \(C \in \mathcal{B}_w(X, Y)\) 
is simply polar if and only if \(C \star C^{\sigma,W} = C\). From the equation

\[
C \star C^{\sigma,W} = CW(CW)^D C = (CW)^\circ C
\]

we conclude that the simple polarity of \(C \in \mathcal{B}_w(X, Y)\) is equivalent to \((CW)^\circ C = C\).

Finally, \(r_w(U) = r(UW)\), and \(UW\) is quasinilpotent in \(\mathcal{B}_w(X, Y)\) if and only if \(U\) 
is quasinilpotent in \(B_w(X, Y)\). This proves the equivalence of \((A)\) and \((B)\) to the 
\(Wg\)-Drazin invertibility of \(A\). Explicitly, \(C = A \star A^{\circ,W} = (AW)^\circ A\).

Towards \((B)\) in view of Theorem 2.4,

\[
(AW)^D = ((AW)^D)^2AW = A^{D,W}W = C^{D,W}W = ((CW)^D)^2CW = (CW)^D.
\]

Therefore

\[
(CW)^\circ = (CW)^D C = (AW)^D (AW)^\circ A = (AW)^D A = (AW)^\circ.
\]

If \(\text{Sp}_w(A)\) denotes the spectrum of \(A\) as an element of the Banach algebra \(B_w(X, Y)\) 
without unit, then it can be shown that \(\text{Sp}_w(A) = \text{Sp}(AW) \cup \{0\}\). Hence

\[
\text{Sp}(AW) \cup \{0\} = \text{Sp}_w(A) = \text{Sp}_w(C) = \text{Sp}(CW) \quad (\text{as } 0 \in \text{Sp}(CW)).
\]

This completes the proof.
The statement of the theorem remains true when (4.3) is replaced by \( C(WC)^\circ = C \), \( WU \in B(X)^{g_{\mathfrak{nil}}} \), and \( AW, CW \) in (4.4) are replaced by \( WA, WC \), respectively.

We close the section with an integral representation of the \( Wg \)-Drazin inverse. The representation of the \( g \)-Drazin inverse given by Castro et al. [6, Theorem 2.2] is valid also for Banach algebras without unit. Applying this result to \( B_w(X, Y) \), we get the integral representation

\[
A^{D, W} = - \int_0^\infty \exp(tA) \star A^{\sigma, W} dt
\]

provided \( A \) is \( Wg \)-Drazin invertible and the non-zero spectrum \( \text{Sp}_w(A) \setminus \{0\} \) lies in the open left half-plane. We express \( \exp(tA) \star A^{\sigma, W} \) in terms of the usual multiplication of operators:

\[
A^n \star A^{\sigma, W} = (AW)^{n-1}AWA^{\sigma, W} = (AW)^n A^{\sigma, W}.
\]

Hence

\[
\exp(tA) \star A^{\sigma, W} = \sum_{n=0}^\infty \frac{t^n}{n!} (AW)^n A^{\sigma, W} = \exp(tAW)A^{\sigma, W}.
\]

Note that in general \( \exp(tAW) \) belongs to the unitisation of \( B_w(X, Y) \) but not to \( B_w(X, Y) \); while \( \exp(tAW)A^{\sigma, W} \) is in \( B_w(X, Y) \). We summarise our findings.

**PROPOSITION 4.4.** Let \( A \in B(X, Y) \) be \( Wg \)-Drazin invertible such that \( \text{Sp}(WA) \setminus \{0\} \) lies in the left open half-plane. Then

\[
A^{D, W} = - \int_0^\infty \exp(tAW)A^{\sigma, W} dt.
\]

If the Drazin index \( \text{ind}(AW) \) is finite and the set \( \text{Sp}((AW)^{m+1}) \setminus \{0\} \) lies in the left open half-plane for some \( m \geq \min \{\text{ind}(AW), \text{ind}(WA)\} + 1 \), then

\[
A^{D, W} = - \int_0^\infty \exp(t(AW)^{m+1}) (AW)^{m-1} A dt.
\]

**PROOF.** Equation (4.5) follows from our calculations preceding the theorem. For (4.6) we find

\[
(A^{\star(m+1)})^m \star A^m = A^{\star(m+1)m+m} = (AW)^{(m+1)m} (AW)^{m-1} A,
\]

and

\[
\exp(tA^{\star(m+1)}) \star A^m = \exp(t(AW)^{m+1}) (AW)^{m-1} A.
\]

Equation (4.6) then follows from [6, Theorem 2.4].
As expected from symmetry, there is also a $WA$ version of the preceding theorem. If we specialise Equation (4.6) to matrices, we recover [18, Theorem 1]. The inequality $m \geq \min \{i(AW), i(WA)\} + 1$ in the preceding theorem can be relaxed to $m \geq \iota_W(A)$.

Using the core decomposition of a $Wg$-Drazin invertible operator $A \in \mathcal{B}(X, Y)$, we obtain yet another integral representation for $A^{D,W}$.

**COROLLARY 4.5.** Let $A \in \mathcal{B}(X, Y)$ be $Wg$-Drazin invertible such that $\text{Sp}(W^2) \setminus \{0\}$ lies in the open left half-plane, and let $A = C + U$ be the core decomposition of $A$. Then

$$A^{D,W} = C^{D,W} = -\int_0^\infty \exp(tCW^2)C \, dt.$$  

**PROOF.** This follows from (4.6) when we note that $\iota_W(C) = 1$. \hfill $\Box$

5. Ascent and descent

We recall that the ascent and descent of an operator $T \in \mathcal{B}(X)$ are defined by

$$\text{asc}(T) = \inf \{ k \in \mathbb{N} : N(T^{k+1}) = N(T^k) \},$$

$$\text{des}(T) = \inf \{ k \in \mathbb{N} : R(T^{k+1}) = R(T^k) \}.$$  

([inf $\emptyset = \infty$). Rakočević and Wei [13] ask whether the finiteness of $\text{asc}(AW)$ and $\text{des}(WA)$ is sufficient for $A$ to have the $W$-weighted Drazin inverse. An equivalent question is whether $\text{asc}(AW)$ and $\text{asc}(WA)$ are always both finite or both infinite.

In this connection it is interesting to recall that Buoni and Faires [3] studied the ascent and descent for the operators $\lambda I - BA$ and $\lambda I - AB$, where $A, B \in \mathcal{B}(X)$, and proved, *inter alia*, that for any $\lambda \neq 0$,

$$\text{asc}(AB - \lambda I) = \text{asc}(BA - \lambda I), \quad \text{des}(AB - \lambda I) = \text{des}(BA - \lambda I);$$

(5.1) however, the case $\lambda = 0$ was left open. Later, Barnes [1] proved by different methods that the ascents of $I - RS$ and $I - SR$ are equal for $R \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, X)$. It can be shown that the arguments in [3] concerning descent are valid also when $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$. Thus (5.1) is valid for operators between different spaces. The following theorem, dealing with the ascent and descent in general, completes the results of Buoni and Faires in the case $\lambda = 0$.

**THEOREM 5.1.** Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$. Then the ascents (descents) of $AB$ and $BA$ are both finite or both infinite, and satisfy the inequalities

$$\text{asc}(AB) - 1 \leq \text{asc}(BA) \leq \text{asc}(AB) + 1,$$

$$\text{des}(AB) - 1 \leq \text{des}(BA) \leq \text{des}(AB) + 1.$$  

(5.2)
PROOF. Suppose that $\text{asc}(AB) = p < \infty$. If there existed
$$x \in N((BA)^{p+2}) \setminus N((BA)^{p+1}),$$
we would have $(AB)^{p+2}Ax = A(BA)^{p+2}x = 0$, and $B(AB)^{p}Ax = (BA)^{p+1}x \neq 0$, that is, $(AB)^{p}Ax \neq 0$. Then $Ax$ would belong to $N((AB)^{p+2}) \setminus N((AB)^{p})$, which is empty by assumption. This contradiction proves $N((BA)^{p+1}) = N((BA)^{p+2})$, which shows that $\text{asc}(BA) \leq p + 1 = \text{asc}(AB) + 1$. A symmetrical argument gives $\text{asc}(AB) \leq \text{asc}(BA) + 1$. This proves the first inequality in (5.2).

Let $\text{des}(AB) = p < \infty$. Suppose
$$x \in R((BA)^{p+1}) \setminus R((BA)^{p+2}).$$
Then there exists $x' \in X$ such that
$$x = (BA)^{p+1}x' = B(AB)^{p}Ax' = By,$$
where $y = (AB)^{p}Ax' \in R((BA)^{p}) = R((AB)^{p+2})$. Hence $y = (AB)^{p+2}y'$ for some $y' \in Y$, and $(BA)^{p+2}y' = B(AB)^{p+2}y' = By = x$ contrary to (5.3). This proves that $R((BA)^{p+1}) = R((BA)^{p+2})$, so that $\text{des}(BA) \leq p + 1$.

The inequalities in (5.2) can be strict; this follows from Example 3.5 since for matrices $i(AB) = \text{asc}(AB) = \text{des}(AB)$.

The following theorem gives a solution to the open problem of Rakočević and Wei [13, page 28].

**Theorem 5.2.** Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X) \setminus \{0\}$. Then $A$ is W-Drazin invertible if and only if one of the following equivalent conditions hold:

(i) $AW$ is polar in $\mathcal{B}(Y)$;
(ii) $WA$ is polar in $\mathcal{B}(X)$;
(iii) $\text{asc}(AW)$ and $\text{des}(WA)$ are both finite;
(iv) $\text{asc}(WA)$ and $\text{des}(AW)$ are both finite.

**Proof.** Suppose that $A$ is W-Drazin invertible. By Theorem 3.3, $AW$ is quasipolar, and by (3.9) we have $i(\text{AW}) \leq i_{W}(A)$, which proves that $AW$ is polar. Conversely, if $AW$ is polar, then $i_{W}(A) \leq i(\text{AW}) + 1$, and $A$ is W-Drazin invertible.

(i) implies (ii): Since $AW$ is quasipolar, so is $WA$ by (3.10). By (3.9) again, $i(WA) \leq i(\text{AW}) + 1$, and $WA$ is polar.

(ii) implies (iii): It is well known that if $WA$ is polar, then $\text{asc}(WA)$ and $\text{des}(WA)$ are both finite. However, $\text{asc}(AW) \leq \text{asc}(WA) + 1$ by Theorem 5.1 and (iii) follows.

(iii) implies (iv): This follows from Theorem 5.1 as $\text{asc}(WA) \leq \text{asc}(AW) + 1$ and $\text{des}(AW) \leq \text{des}(WA) + 1$.

(iv) implies (i): Since $\text{asc}(AW) \leq \text{asc}(WA) + 1$, both $\text{asc}(AW)$ and $\text{des}(AW)$ are finite; this implies that $AW$ is polar. 

\[\boxed{\text{\(AW\) is polar \iff \text{asc}(\text{AW}) \leq \text{asc}(\text{WA}) + 1 \text{ and } \text{des}(\text{AW}) \leq \text{des}(\text{WA}) + 1\)}}\]
6. The Mbekhta decomposition for WA and AW

As before, \(X, Y\) are Banach spaces and \(W\) a nonzero operator in \(\mathcal{B}(Y, X)\). In order to obtain an operator matrix representation for the weighted \(g\)-Drazin inverse of an operator \(A \in \mathcal{B}(X, Y)\), we first recall the Mbekhta decomposition for a quasipolar operator. For any operator \(T \in \mathcal{B}(X)\) we define spaces \(H_0(T)\) and \(K(T)\) as follows:

\[
H_0(T) = \left\{ x \in X : \lim_{n \to \infty} \|T^n x\|^{1/n} = 0 \right\},
\]

\[
K(T) = \left\{ x \in X : \exists x_n \in X, \quad x_n = T x_{n+1}, \quad x_0 = x, \quad \sup_{n \in \mathbb{N}} \|x_n\|^{1/n} < \infty \right\}.
\]

Both spaces are hyperinvariant under \(T\), \(H_0(T) \supset N(T^*)\), and \(K(T) \subset R(T^*)\) for all \(n \in \mathbb{N}\). Further, \(TK(T) = K(T)\) and \(T^{-1} H_0(T) = H_0(T)\).

**PROPOSITION 6.1** (See [8, 11]). The following conditions on \(T \in \mathcal{B}(X)\) are equivalent:

(i) \(T\) is quasipolar;

(ii) \(X\) is the topological direct sum \(X = K(T) \oplus H_0(T)\);

(iii) \(T = T_1 \oplus T_2\), where \(T_1\) is invertible and \(T_2\) quasinilpotent.

Condition (ii) can be weakened to \(X = K(T) \oplus H_0(T)\) being only an algebraic sum with at least one of the spaces closed (see [10] and [15]).

**THEOREM 6.2.** Let \(A \in \mathcal{B}(X, Y)\) and \(W \in \mathcal{B}(Y, X)\). If \(WA\) is quasipolar, then so is \(AW\),

\[
A(K(WA)) = K(AW), \quad A^{-1}(H_0(AW)) = H_0(WA),
\]

\[
W(K(AW)) = K(WA), \quad W^{-1}(H_0(WA)) = H_0(AW),
\]

and the spaces \(K(WA), K(AW)\) are isomorphic and homeomorphic.

**PROOF.** The result on quasipolarity follows from (3.10). We introduce the following notation

\[
(6.1) \quad X_1 = K(WA), \quad X_2 = H_0(WA), \quad Y_1 = K(AW), \quad Y_2 = H_0(AW).
\]

Then \(X\) and \(Y\) are decomposed into the topological direct sums \(X = X_1 \oplus X_2\) and \(Y = Y_1 \oplus Y_2\). The operator matrices

\[
(6.3) \quad T = \begin{bmatrix} WA & 0 \\ 0 & AW \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}
\]
represent commuting operators in $\mathcal{B}(X \oplus Y)$ with $T$ quasipolar. The support projection $T^\sigma$ of $T$ double commutes with $T$, that is, the matrix

$$T^\sigma = \begin{bmatrix} (WA)^\sigma & 0 \\ 0 & (AW)^\sigma \end{bmatrix}$$

commutes with the matrix $S$. This gives $A(WA)^\sigma = (AW)^\sigma A$. Since $(WA)^\sigma$ is the projection of $X$ onto $X_1$ along $X_2$, and $(AW)^\sigma$ is the projection of $Y$ onto $Y_1$ along $Y_2$, we have $A(X_i) \subset Y_i$ ($i = 1, 2$). The inclusions $W(Y_i) \subset X_i$ ($i = 1, 2$) are obtained by symmetry.

Note that $A(X_2) \subset Y_2$ is equivalent to $X_2 \subset A^{-1}(Y_2)$. In order to prove $A^{-1}(Y_2) \subset X_2$, assume that $Ax \in Y_2$. Then $x = k + h$ with $k \in X_1$ and $h \in X_2$, and $Ax = Ak + Ah \in Y_2$ implies that $Ak = 0$. From $k \in N(A) \subset N(WA) \subset X_2$, we obtain $k \in X_1 \cap X_2 = \{0\}$. Hence $x = h \in X_2$.

Let $A_0 : X_1 \to Y_1$ be the restriction of $A$. If $x \in X_1$ and $Ax = 0$, then $x = 0$ by the argument of the preceding paragraph. Hence $A_0$ is injective. Suppose that $y \in Y_1$. Since $AWY_1 = Y_1$, there exists $u \in Y_1$ such that $y = AWu$. But $WY_1 \subset X_1$, and so $Wu \in X_1$. This proves that $A_0$ is surjective. Therefore $A_0$ is a bounded linear bijection from $X_1$ to $Y_1$, and (6.1) is proved.

In particular, if $AW$ is quasipolar, then the spaces $K(AW)$ and $K(WA)$ have the same dimension being isomorphic.

If $A$ and $W$ are rectangular matrices of orders $m \times n$ and $n \times m$ respectively, we recover the result of Yukhno [19, Theorem]. For this the operator $T : \mathbb{C}^n \to \mathbb{C}^m$ with the matrix $WA$ is polar, and $T = T_1 \oplus T_2$, where $T_1$ is invertible and $T_2$ nilpotent; $T_1$ operates on $X_1 = K(T) = R(T^p)$, where $p$ is the index of $T$. The eigenvalues of $T_1$ are the nonzero eigenvalues of $WA$. Let $\lambda$ be a nonzero eigenvalue of $WA$, and $x_1, \ldots, x_k$ a chain of generalised eigenvectors of $WA$ corresponding to $\lambda$, that is,

$$WAx_1 = \lambda x_1 + x_2, \quad \ldots, \quad WAx_{k-1} = \lambda x_{k-1} + x_k, \quad WAx_k = \lambda x_k.$$ 

In view of the decomposition of $T$ as $T = T_1 \oplus T_2$, where $T_1$ operates on $X_1$, we can take $x_i \in X_1$ for all $i$. If $y_i = Ax_i$ ($i = 1, \ldots, k$), then $y_1, \ldots, y_k$ is a chain of generalised eigenvectors of $AW$ corresponding to $\lambda$ (this follows from the bijectivity of the operator $x \mapsto Ax$ restricted from $X_1$ to $Y_1$). All chains corresponding to nonzero eigenvalues of $WA$ are matched in this way. This leads to the following structure theorem for $WA$ and $AW$.

**Proposition 6.3.** Let $A$ and $W$ be rectangular matrices of orders $m \times n$ and $n \times m$, respectively. The matrices $WA$ and $AW$ (of orders $n \times n$ and $m \times m$, respectively) have Jordan forms

$$\begin{bmatrix} U & 0 \\ 0 & N_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & 0 \\ 0 & N_2 \end{bmatrix},$$

where $U$ is invertible and $N_1$ and $N_2$ are nilpotent matrices.
where $U$ is a matrix in Jordan form corresponding to the nonzero eigenvalues of $WA$ (and $AW$), while $N_1$ and $N_2$ are nilpotent matrices in Jordan form, of different orders in general.

Recall that the entries of $N_1$ and $N_2$ are zero except for the superdiagonals, which consist of 0s and 1s.

### 7. An operator matrix representation of the $Wg$-Drazin inverse

From the Mbekhta decomposition theorem (Proposition 6.1), it follows that an operator $T \in \mathcal{B}(X)$ is quasipolar if and only if it can be expressed as the direct sum $T = T_1 \oplus T_2$, where $T_1$ is invertible and $T_2$ quasinilpotent; the $g$-Drazin inverse of $T$ is given by

$$T^D = T_1^{-1} \oplus 0.$$

Our aim is to derive an analogous formula for the $Wg$-Drazin inverse using the results of the preceding section.

**Theorem 7.1.** Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X) \setminus \{0\}$. Then $A$ is $Wg$-Drazin invertible if and only if there exist topological direct sums $X = X_1 \oplus X_2, Y = Y_1 \oplus Y_2$ such that $A = A_1 \oplus A_2$ and $W = W_1 \oplus W_2$, where $A_i \in \mathcal{B}(X_i, Y_i), W_i \in \mathcal{B}(Y_i, X_i)$, with $A_1, W_1$ invertible, and $W_2 A_2$ and $A_2 W_2$ quasinilpotent in $\mathcal{B}(X_2)$ and $\mathcal{B}(Y_2)$, respectively. The $Wg$-Drazin inverse of $A$ is given by $A^{D,W} = (W_1 A_1 W_1)^{-1} \oplus 0$ with $(W_1 A_1 W_1)^{-1} \in \mathcal{B}(X_1, Y_1)$ and $0 \in \mathcal{B}(X_2, Y_2)$.

**Proof.** If $WA$ is quasipolar, the decomposition exists with $X_i$ and $Y_i$ given by (6.2). By Theorem 6.2, $A$ maps $X_1$ into $Y_1$, and $X_2$ into $Y_2$, that is, $A = A_1 \oplus A_2$, with $A_i \in \mathcal{B}(X_i, Y_i), i = 1, 2$. Similarly, since $W$ maps $Y_1$ into $X_1$ and $Y_2$ into $X_2$, $W = W_1 \oplus W_2$, where $W_i \in \mathcal{B}(Y_i, X_i), i = 1, 2$. Hence

$$WA = W_1 A_1 \oplus W_2 A_2, \quad AW = A_1 W_1 \oplus A_2 W_2$$

relative to $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Since $WA$ and $AW$ are quasipolar, $W_1 A_1$ and $A_1 W_1$ are invertible, and $W_2 A_2$ and $A_2 W_2$ are quasinilpotent. Hence $A_1$ and $W_1$ are invertible.

The $Wg$-Drazin inverse of $A$ is equal to

$$A((WA)^D)^2 = (A_1 \oplus A_2)((W_1 A_1)^{-2} \oplus 0) = (W_1 A_1 W_1)^{-1} \oplus 0.$$

Conversely, if the decompositions with the specified properties exist, then $AW = (A_1 W_1) \oplus (A_2 W_2)$ is quasipolar as $A_1 W_1$ is invertible and $A_2 W_2$ quasinilpotent. Then $A$ is $Wg$-Drazin invertible.

$\square$
From the necessary part of the preceding theorem we recover [18, Theorem 2] when we specialise the operators to finite matrices. From Theorem 6.2 applied to finite matrices we deduce that the ranks of \((AW)^m\) and \((WA)^m\) are equal for any \(m \geq \max\{\text{nd}(AW), \text{nd}(WA)\}\). (This is used, but not proved, in the derivation of [18, Theorem 2]).

From the commutativity of the operator matrices given in (6.3) and the double commutativity of the \(g\)-Drazin inverse we deduce that \((AW)^D A = A(WA)^D\), which leads to the new equality for \(A^{D,W}\) derived from (3.6),

\[
A^{D,W} = (AW)^D A(WA)^D.
\]

**8. Relation to the Moore–Penrose inverse**

We briefly address the relation of the \(Wg\)-Drazin inverse to the Moore–Penrose inverse in Hilbert spaces (see [13, page 28]). Let \(H, K\) be Hilbert spaces and let \(A \in \mathcal{B}(H, K)\). It is well known that \(R(A)\) is closed if and only if \(R(A^*)\) is closed, \(R(A^*)\) is closed if and only if \(A^*A\) is simply polar, and \(A^*A\) is simply polar if and only if \(AA^*\) is simply polar. This means that \(A \in \mathcal{B}(H, K)\) is \(A^*g\)-Drazin invertible if and only if the range of \(A\) is closed. We note that

\[
(A^{D,A^*})^* = (A^*)^{D,A^*}.
\]

We can then prove that the operator \(A^\dagger = (A^*)^{g,A} = A^* A^{D,A^*} A^*\) is the Moore–Penrose inverse characterised by the equations

\[
A^\dagger AA^\dagger = A^\dagger, \quad AA^\dagger A = A, \quad (A^\dagger A)^* = A^\dagger A, \quad (AA^\dagger)^* = AA^\dagger.
\]

We offer a sample of such proof

\[
A^\dagger AA^\dagger = (A^*)^{g,A} A (A^*)^{g,A} = (A^*)^{g,A} \circ (A^*)^{g,A} = (A^*)^{g,A} = A^\dagger,
\]

where \(T \circ S = TAS\), and

\[
AA^\dagger A = AA^* A^{D,A^*} A^* A = A \ast A^{D,A^*} \ast A = A,
\]

where \(T \ast S = TA^* S\). Other equations in (8.2) can be proved similarly.

**9. Continuity of the \(Wg\)-Drazin inverse**

**Theorem 9.1.** Let \(A_n \to A_0\) in \(\mathcal{B}(X, Y)\) and \(W_n \to W_0 \neq 0\) in \(\mathcal{B}(Y, X)\), where each \(A_n\) is \(W_n g\)-Drazin invertible, \(n = 0, 1, 2, \ldots\) Then the following conditions are equivalent:
(i) \( A_{n}^{D,W_{n}} \to A_{0}^{D,W_{0}} \);
(ii) \( \sup_{n} \| A_{n}^{D,W_{n}} \| < \infty \);
(iii) \( (A_{n}W_{n})^{D} \to (A_{0}W_{0})^{D} \);
(iv) \( A_{n}^{\varepsilon,W_{n}} \to A_{0}^{\varepsilon,W_{0}} \).

**Proof.** We rely on continuity results for the \( g \)-Drazin inverse obtained in [9].

Condition (i) clearly implies (ii). Suppose that (ii) holds. Since

\[
(A_{n}W_{n})^{D} = ((A_{n}W_{n})^{D})^{2}(A_{n}W_{n}) = A_{n}^{D,W_{n}}W_{n},
\]

we have \( \sup_{n} \| (A_{n}W_{n})^{D} \| < \infty \). By [9, Theorem 2.4], \( (A_{n}W_{n})^{D} \to (A_{0}W_{0})^{D} \).

If (iii) holds, then \( A_{n}^{\varepsilon,W_{n}} = (A_{n}W_{n})^{D}A_{n} \to (A_{0}W_{0})^{D}A_{0} = A_{0}^{\varepsilon,W_{0}} \).

Let (iv) hold. From the equation

\[
(A_{n}W_{n})^{\varepsilon} = (A_{n}W_{n})^{D}A_{n}W_{n} = A_{n}^{\varepsilon,W_{n}}W_{n},
\]

we deduce that \( (A_{n}W_{n})^{\varepsilon} \to (A_{0}W_{0})^{\varepsilon} \). Using [9, Theorem 2.4] again, we obtain \( (A_{n}W_{n})^{D} \to (A_{0}W_{0})^{D} \). Hence \( A_{n}^{D,W_{n}} = ((A_{n}W_{n})^{D})^{2}A_{n} \to ((A_{0}W_{0})^{D})^{2}A_{0} = A_{0}^{D,W_{0}} \)

and the theorem is proved.

From the preceding theorem we recover [13, Theorem 5.1] when we specialise the result to a finite index weighted Drazin inverse.

**References**


Department of Mathematics and Statistics
The University of Melbourne
VIC 3010
Australia

e-mail: a.dajic@ms.unimelb.edu.au, j.koliha@ms.unimelb.edu.au