C*-ALGEBRAS ASSOCIATED WITH PRESENTATIONS OF SUBSHIFTS II. IDEAL STRUCTURE AND LAMBDA-GRA PH SUBSYSTEMS

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Abstract

A \(\lambda\)-graph system is a labeled Bratteli diagram with shift transformation. It is a generalization of finite labeled graphs and presents a subshift. In Doc. Math. 7 (2002) 1–30, the author constructed a C*-algebra \(\mathcal{O}_\Sigma\) associated with a \(\lambda\)-graph system \(\Sigma\) from a graph theoretic viewpoint. If a \(\lambda\)-graph system comes from a finite labeled graph, the algebra becomes a Cuntz-Krieger algebra. In this paper, we prove that there is a bijective correspondence between the lattice of all saturated hereditary subsets of \(\mathcal{O}_\Sigma\) and the lattice of all ideals of the algebra \(\mathcal{O}_\Sigma\), under a certain condition on \(\mathcal{O}_\Sigma\) called (II). As a result, the class of the C*-algebras associated with \(\lambda\)-graph systems under condition (II) is closed under quotients by its ideals.


1. Introduction

In [7], Cuntz and Krieger presented a class of C*-algebras associated with finite square matrices with entries in \(\{0, 1\}\). The C*-algebras are called Cuntz-Krieger algebras. They are simple if the matrices are irreducible with condition (I). Cuntz-Krieger observed that the C*-algebras have a close relationship to topological Markov shifts ([7]). The topological Markov shifts form a subclass of subshifts. For a finite set \(\Sigma\), a subshift \((\Lambda, \sigma)\) is a topological dynamical system defined by a closed shift-invariant subset \(\Lambda\) of the compact set \(\Sigma^\mathbb{Z}\) of all bi-infinite sequences of \(\Sigma\) with shift transformation \(\sigma\). In [21] (compare [25, 5]), the author generalized the class of the Cuntz-Krieger algebras to a class of C*-algebras associated with subshifts. He also
introduced several topological conjugacy invariants and presentations for subshifts by using K-theory and algebraic structure of the associated $C^*$-algebras with the subshifts in [23]. For presentation of subshifts, notions of the $\lambda$-graph system and symbolic matrix system have been introduced ([23]). They are generalizations of the $\lambda$-graph (labeled graph) and the symbolic matrix for sofic subshifts to general subshifts.

We henceforth denote by $\mathbb{Z}_+$ the set of all nonnegative integers. Let $\Sigma$ be a finite set that is called an alphabet. A $\lambda$-graph system $\mathcal{L} = (V, E, \lambda, i)$ consists of a vertex set $V = \bigcup_{l \in \mathbb{Z}_+} V_l$, an edge set $E = \bigcup_{l \in \mathbb{Z}_+} E_{l+1}$, a labeling map $\lambda : E \to \Sigma$ and a surjective map $i(= i_{l+1}) : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$ with a certain compatible condition, called the local property. Its matrix presentation $\mathcal{M}_{l+1, l}$ is called a symbolic matrix system, denoted by $(\mathcal{M}, I)$. The $\lambda$-graph systems give rise to subshifts by gathering label sequences appearing in the labeled Bratteli diagrams of the $\lambda$-graph systems. Conversely, there is a canonical method to construct a $\lambda$-graph system from an arbitrary subshift [23]. It is called the canonical $\lambda$-graph system for subshift $\Lambda$.

In [24], the author constructed $C^*$-algebras from $\lambda$-graph systems and studied their structure. Let $\mathcal{L} = (V, E, \lambda, i)$ be a $\lambda$-graph system over alphabet $\Sigma$. Let $\{v^l_1, \ldots, v^l_{m(l)}\}$ be the set of the vertex $V_l$. We henceforth assume that a $\lambda$-graph system $\mathcal{L}$ is left-resolving, that is, there are no distinct edges with the same label and the same terminal vertex. The $C^*$-algebra $\mathcal{O}_\mathcal{L}$ is realized as a universal unique $C^*$-algebra subject to certain operator relations among generating partial isometries $S_\alpha$, corresponding to the symbols $\alpha \in \Sigma$ and projections $E^l_i$ corresponding to the vertices $v^l_i \in V_l$, $i = 1, \ldots, m(l)$, $l \in \mathbb{Z}_+$, encoded by the concatenation rule of $\mathcal{L}$. Irreducibility and aperiodicity for finite directed graphs have been generalized to $\lambda$-graph systems in [24]. If $\mathcal{L}$ satisfies condition (I), a condition generalizing condition (I) for finite square matrices defined by [7], and is irreducible, then the $C^*$-algebra $\mathcal{O}_\mathcal{L}$ is simple. In particular, if $\mathcal{L}$ is aperiodic, then $\mathcal{O}_\mathcal{L}$ is simple and purely infinite ([24], compare [27]).

In this paper, we investigate ideal structures of the $C^*$-algebras $\mathcal{O}_\mathcal{L}$. The discussions are based on a line of Cuntz’s paper [6] in which the ideal structure of the Cuntz-Krieger algebras were studied (compare [13]). We generalize condition (II) for finite directed graphs, defined in [6], to $\lambda$-graph systems. By considering saturated hereditary subsets of $\mathcal{L}$ with respect to arrows of edges, we show the following theorem.

**THEOREM A** (Proposition 3.5, Theorem 3.6). Suppose that $\mathcal{L}$ satisfies condition (II). There is a bijective correspondence between the lattice of all saturated hereditary subsets of $\mathcal{L}$ and the lattice of all ideals of the algebra $\mathcal{O}_\mathcal{L}$. Furthermore, for any ideal $\mathcal{I}$ of $\mathcal{O}_\mathcal{L}$, the quotient $C^*$-algebra $\mathcal{O}_\mathcal{L}/\mathcal{I}$ is isomorphic to the $C^*$-algebra $\mathcal{O}_{\mathcal{L}/\mathcal{I}}$ associated with the $\lambda$-graph system $\mathcal{L}/\mathcal{I}$, obtained by removing the corresponding saturated hereditary subset $C_\mathcal{I}$ for $\mathcal{I}$. 
**Corollary B.** In the $\lambda$-graph systems satisfying condition (II), the class of the $C^*$-algebras associated with $\lambda$-graph systems is closed under quotients by ideals.

By Corollary B, it is expected that rich examples of simple purely infinite nuclear $C^*$-algebras of this class live outside Cuntz-Krieger algebras (compare [24, Theorem 7.7], [16], [26] and [20]). We further study the structure of an ideal of $O_C$ in Section 4. We prove that an ideal of $O_C$ is stably isomorphic to the $C^*$-subalgebra of $O_C$ associated with the corresponding saturated hereditary subset of $V$ (Theorem 4.3). As a result, the K-theory formulae for ideals of $O_C$ are presented in terms of the corresponding saturated hereditary subsets of $V$ (Theorem 4.5).

If a $\lambda$-graph system $\mathcal{L}$ comes from a finite directed graph $G$, the associated $C^*$-algebra $O_C$ becomes a Cuntz-Krieger algebra $O_{\lambda_C}$ for its adjacency matrix $A_G$ with entries in $\{0, 1\}$. The results of this paper, Theorem A, Corollary B, Theorem 4.3, Theorem 4.5, and Proposition 4.6 are generalizations of Cuntz’s result [6, Theorem 2.5] for Cuntz-Krieger algebras. Other generalizations of Cuntz-Krieger algebras from this graph point of view have been studied by [2, 10, 12, 15, 17, 18, 30, 34] and [35]. Related discussions for $C^*$-algebras generated by Hilbert $C^*$-bimodules can be found in [14].

2. Review of the $C^*$-algebras associated with $\lambda$-graph systems

Recall that a $\lambda$-graph system $\mathcal{L} = (V, E, \lambda, \iota)$ over an alphabet $\Sigma$ is a directed Bratteli diagram with vertex set $V = \bigcup_{l \in \mathbb{Z}_+} V_l$ and edge set $E = \bigcup_{l \in \mathbb{Z}_+} E_{l,l+1}$ that is labeled with symbols in $\Sigma$ by $\lambda : E \to \Sigma$, and that is supplied with surjective maps $\iota(= \iota_{l,l+1}) : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$. Here, both the vertex sets $V_l$, $l \in \mathbb{Z}_+$ and the edge sets $E_{l,l+1}$, $l \in \mathbb{Z}_+$ are finite disjoint sets. An edge $e$ in $E_{l,l+1}$ has its source vertex $s(e)$ in $V_l$ and its terminal vertex $t(e)$ in $V_{l+1}$ respectively. Every vertex in $V$ has a successor and every vertex in $V_l$ for $l \in \mathbb{N}$ has a predecessor. It is required that there exists a bijective correspondence, which preserves labels, between $\{e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u\}$ and $\{e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v)\}$ for all pairs of vertices $u \in V_{l-1}$ and $v \in V_{l+1}$. This property of the $\lambda$-graph systems is called the local property. We call an edge $e \in E_{l,l+1}$ a $\lambda$-edge and a connecting finite sequence of $\lambda$-edges a $\lambda$-path. For $u, v \in V$, if $\iota(v) = u$, we say that there exists an $\iota$-edge from $v$ to $u$. Similarly we use the term $\iota$-path. We denote by $\{v'_1, v'_2, \ldots, v'_m\}$ the vertex set $V_l$ of $V$ at level $l$. A finite labeled graph $(G, \lambda)$ over $\Sigma$ yields a $\lambda$-graph system $\mathcal{L}_{(G,\lambda)}$ by setting $V_l = V_l$, $E_{l,l+1} = E$ for $l \in \mathbb{Z}_+$ and $\iota = \text{id}$ (compare [24, Section 7]).

Let us now briefly review the $C^*$-algebra $O_C$ associated with the $\lambda$-graph system $\mathcal{L}$, which was originally constructed in [24] to be a groupoid $C^*$-algebra of a groupoid
of a continuous graph obtained by \( \mathcal{L} \) (compare \([8, 9, 31]\)). The \( C^* \)-algebras \( \mathcal{O}_\mathcal{L} \) are generalization of the \( C^* \)-algebras associated with subshifts. That is, if the \( \lambda \)-graph system is the canonical \( \lambda \)-graph system for a subshift \( \Lambda \), the constructed \( C^* \)-algebra coincides with the \( C^* \)-algebra \( \mathcal{O}_\Lambda \) associated with the subshift \( \Lambda \) in \([26]\) (compare \([5]\)).

Let \( \mathcal{L} = (V, E, \lambda, i) \) be a left-resolving \( \lambda \)-graph system over \( \Sigma \). We denote by \( \Lambda \) the presented subshift \( \Lambda_\mathcal{L} \) by \( \mathcal{L} \). We denote by \( \Lambda^\lambda \) the set of admissible words in \( \Lambda \) of length \( k \). We set \( \Lambda^\lambda = \bigcup_{k=0}^{\infty} \Lambda^k \), where \( \Lambda^0 \) denotes the empty word. Define the transition matrices \( A_{i,j+1}, I_{i,j+1} \) of \( \mathcal{L} \) by setting for \( i = 1, 2, \ldots, m(l), j = 1, 2, \ldots, m(l+1), \alpha \in \Sigma \),

\[
A_{i,j+1}(i, \alpha, j) = \begin{cases} 
1 & \text{if } s(e) = v^i_j, \lambda(e) = \alpha, t(e) = v^{i+1}_j \text{ for some } e \in E_{i,j+1}, \\
0 & \text{otherwise},
\end{cases}
\]

\[
I_{i,j+1}(i, j) = \begin{cases} 
1 & \text{if } i_{i,j+1}(v^{i+1}_j) = v^i_j, \\
0 & \text{otherwise}.
\end{cases}
\]

The \( C^* \)-algebra \( \mathcal{O}_\mathcal{L} \) is realized as the universal unital \( C^* \)-algebra generated by partial isometries \( S_\alpha, \alpha \in \Sigma \) and projections \( E_i^l, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+ \), subject to the following operator relations called \( (\mathcal{L}) \)

\[
\sum_{\alpha \in \Sigma} S_\alpha S_\alpha^* = 1, \tag{2.1}
\]

\[
\sum_{i=1}^{m(l)} E_i^l = 1, \quad E_i^l = \sum_{j=1}^{m(l+1)} I_{i,j+1}(i, j) E_j^{i+1}, \tag{2.2}
\]

\[
S_\beta S_\beta^* E_i^l = E_i^l S_\beta S_\beta^*, \tag{2.3}
\]

\[
S_\beta^* E_i^l S_\beta = \sum_{j=1}^{m(l+1)} A_{i,j+1}(i, \beta, j) E_j^{i+1}, \tag{2.4}
\]

for \( \beta \in \Sigma, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+ \). It is nuclear ([24, Proposition 5.6]). The relations \( (2.1), (2.3) \) and \( (2.4) \) yield the relations

\[
E_i^l = \sum_{\alpha \in \Sigma} \sum_{j=1}^{m(l+1)} A_{i,j+1}(i, \alpha, j) S_\alpha E_j^{i+1} S_\alpha^*, \tag{2.5}
\]

for \( i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+ \). For a word \( \mu = \mu_1 \cdots \mu_k \in \Lambda^k \), we set \( S_\mu = S_{\mu_1} \cdots S_{\mu_k} \). Then the algebra of all finite linear combinations of the elements of the form \( S_\mu E_i^l S_\nu \), for \( \mu, \nu \in \Lambda^*, i = 1, \ldots, m(l), l \in \mathbb{Z}_+ \), is a dense \(*\)-subalgebra of \( \mathcal{O}_\mathcal{L} \).

We define three \( C^* \)-subalgebras \( \mathcal{F}_l^k \), \( \mathcal{F}_l^\infty \) and \( \mathcal{F}_l \) of \( \mathcal{O}_\mathcal{L} \). The first one, \( \mathcal{F}_l^k \), is generated by \( S_\mu E_i^l S_\nu \), \( \mu, \nu \in \Lambda^k, i = 1, \ldots, m(l) \), the second one, \( \mathcal{F}_l^\infty \), is
generated by \( \mathcal{F}_k^l, k \leq l, l \in \mathbb{Z}_+ \), and the third one, \( \mathcal{F}_E \), is generated by \( \mathcal{F}_k^\infty, k \in \mathbb{Z}_+ \). There exist two embeddings \( \iota_{l+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1} \), coming from the second relation of (2.2) and \( \lambda_{k,k+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1} \), coming from (2.5). The latter embeddings induce an embedding of \( \mathcal{F}_k^\infty \) into \( \mathcal{F}_{k+1}^\infty \) that we also denote by \( \lambda_{k,k+1} \). Since the algebra \( \mathcal{F}_k^l \) is finite dimensional, the embeddings \( \iota_{l+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1}, l \in \mathbb{N} \) yield the AF-algebra \( \mathcal{F}_k^\infty \), and the embeddings \( \lambda_{k,k+1} : \mathcal{F}_k^\infty \hookrightarrow \mathcal{F}_{k+1}^\infty, k \in \mathbb{N} \) yield the AF-algebra \( \mathcal{F}_E \).

For a vertex \( v_i \in V_l \), set

\[
\Gamma^+(v_i) = \left\{ (\alpha_1, \alpha_2, \ldots) \in \Sigma^\infty \mid \text{there exists an edge } e_{n,n+1} \in E_{n,n+1} \text{ for } n \geq l \text{ such that } v_i^l = s(e_{n,n+1}), t(e_{n,n+1}) = s(e_{n+1,n+2}), \lambda(e_{n,n+1}) = \alpha_{n-l+1} \right\},
\]

the set of all label sequences in \( \Sigma \) starting at \( v_i \). We say that \( \Sigma \) satisfies condition (I) if for each \( v_i \in V \), the set \( \Gamma^+(v_i) \) contains at least two distinct sequences. Under condition (I), the algebra \( \mathcal{O}_\Sigma \) can be realized as the unique C*-algebra subject to the relations (\( \Sigma \)). This means that if \( \hat{S}_\alpha, \alpha \in \Sigma \), and \( \hat{E}_i, i = 1, \ldots, m(l), l \in \mathbb{Z}_+ \), are another family of nonzero partial isometries and nonzero projections satisfying the relations (\( \Sigma \)), then the map \( \hat{S}_\alpha \to \hat{S}_\alpha, \hat{E}_i \to \hat{E}_i \) extends to an isomorphism from \( \mathcal{O}_\Sigma \) onto the C*-algebra \( \hat{\mathcal{O}}_\Sigma \) generated by \( \hat{S}_\alpha, \alpha \in \Sigma \), and \( \hat{E}_i, i = 1, \ldots, m(l), l \in \mathbb{Z}_+ \) ([24, Theorem 4.3]).

Let \( \mathcal{A}_\Sigma \) be the C*-subalgebra of \( \mathcal{O}_\Sigma \) generated by the projections \( E_i, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+ \). Let \( \Omega_\Sigma \) the projective limit of the system \( \iota_{l+1} : V_{l+1} \to V_l, l \in \mathbb{Z}_+ \). We endow \( \Omega_\Sigma \) with the projective limit topology so that it is a compact Hausdorff space. An element of \( \Omega_\Sigma \) is called an \( \iota \)-orbit. By the universality of the algebra \( \mathcal{O}_\Sigma \), the algebra \( \mathcal{A}_\Sigma \) is isomorphic to the commutative C*-algebra \( \mathcal{C}(\Omega_\Sigma) \) of all complex valued continuous functions on \( \Omega_\Sigma \). As a corollary of [24, Theorem 4.3], if \( \Sigma \) satisfies condition (I), for a nonzero ideal \( \mathcal{I} \) of \( \mathcal{O}_\Sigma \), we have \( \mathcal{I} \cap \mathcal{A}_\Sigma \neq \{0\} \).

A \( \lambda \)-graph system \( \Sigma \) is said to be irreducible if for a vertex \( v \in V \) and an \( \iota \)-orbit \( x = (x_t)_{t \in \mathbb{Z}_+} \in \Omega_\Sigma \), there exists a \( \lambda \)-path starting at \( v \) and terminating at \( x_{t+N} \) for some \( N \in \mathbb{N} \). Define a positive operator \( \lambda_\Sigma \) on \( \mathcal{A}_\Sigma \) by \( \lambda_\Sigma(X) = \sum_{\alpha \in \Sigma} S_\alpha X S_\alpha \) for \( X \in \mathcal{A}_\Sigma \). The operator \( \lambda_\Sigma \) on \( \mathcal{A}_\Sigma \) induces the embedding \( \mathcal{F}_k^\infty \subset \mathcal{F}_{k+1}^\infty, k \in \mathbb{N} \) so as to define the AF-algebra \( \mathcal{F}_\Sigma = \lim_{\leftarrow k} \mathcal{F}_k^\infty \). We say that \( \lambda_\Sigma \) is irreducible if there exists no non-trivial ideal of \( \mathcal{A}_\Sigma \) invariant under \( \lambda_\Sigma \). Then \( \Sigma \) is irreducible if and only if \( \lambda_\Sigma \) is irreducible. If \( \Sigma \) is irreducible with condition (I), the C*-algebra \( \mathcal{O}_\Sigma \) is simple ([24, Theorem 4.7], compare [27]).

### 3. Hereditary subsets of the vertices and ideals

This section and the next section are the main parts of this paper. In what follows we assume that a \( \lambda \)-graph system \( \Sigma = (V, E, \lambda, \iota) \) over \( \Sigma \) is left-resolving and satisfies
condition (I). We mean by an ideal of a $C^*$-algebra a closed two-sided ideal. Recall that the vertex set $V_i$ is denoted by $\{v_i^j, \ldots, v_{m_i}^j\}$. For $v_i^j \in V_i$ and $v_{i+1}^{j+1} \in V_{i+1}$, we write $v_i^j \geq v_{i+1}^{j+1}$ if $u_{i+1}(v_{i+1}^{j+1}) = v_i^j$. We also write $v_i^j \geq v_{i+1}^{j+1}$ if there exists an edge $e \in E_{i+1}$ such that $s(e) = v_i^j, t(e) = v_{i+1}^{j+1}$. For $v_i^j \in V_i$ and $v_{i+k}^m \in V_{i+k}$, we write $v_i^j \geq v_{i+k}^m$ (respectively $v_i^j \geq v_{i+k}^m$) if there exist $v_{i+1}^{j+1}, \ldots, v_{i+k+1}^{j+k+1}$ such that $v_i^j \geq v_{i+1}^{j+1} \geq \cdots \geq v_{i+k+1}^{j+k+1} \geq v_{i+k}^m$ (respectively $v_i^j \geq v_{i+1}^{j+1} \geq \cdots \geq v_{i+k}^{j+k} \geq v_{i+k+1}^{j+k+1}$).

A subset $C$ of $V$ is said to be $i$-hereditary (respectively $\lambda$-hereditary) if for $v_i^j \in C \cap V_i$ the condition $v_i^j \geq v_{i+1}^{j+1}$ (respectively $v_i^j \geq v_{i+1}^{j+1}$) implies $v_{i+1}^{j+1} \in C$. It is said to be hereditary if $C$ is both $i$-hereditary and $\lambda$-hereditary. It is said to be $i$-saturated (respectively $\lambda$-saturated) if it contains every vertex $v_i^j \in C \cap V_i$ for which $v_i^j \geq v_{i+1}^{j+1}$ (respectively $v_i^j \geq v_{i+1}^{j+1}$) implies $v_{i+1}^{j+1} \in C$. If $C$ is both $i$-saturated and $\lambda$-saturated, it is said to be saturated.

**Definition.** A $\lambda$-graph system $\Sigma' = (V', E', \lambda', i')$ over $\Sigma'$ is said to be a $\lambda$-graph subsystem of $\Sigma$ if it satisfies the following conditions:

$$\emptyset \neq V'_i \subset V_i, \quad \emptyset \neq E'_{i+1} \subset E_{i+1}, \quad \text{for } i \in \mathbb{Z}_+,$$

$$\lambda|_{E'} = \lambda', \quad i|_{V'} = i', \quad \Sigma' \subset \Sigma,$$

and an edge $e \in E$ belongs to $E'$ if and only if the both vertices $s(e), t(e)$ belong to $V'$. Hence a $\lambda$-graph subsystem is determined by only its vertex set.

**Lemma 3.1.** For a saturated hereditary subset $C \subset V$, set

$$V'^C = V \setminus C,$$

$$E'^C = \{e \in E \mid s(e), t(e) \in V \setminus C\},$$

$$\lambda'^C = \lambda|_{E'^C}, \quad i'^C = i|_{V'^C}.$$

Then $(V'^C, E'^C, \lambda'^C, i'^C)$ is a $\lambda$-graph subsystem over $\Sigma$ of $\Sigma$.

**Proof.** For a vertex $u \in V'^C$, there exists a vertex $w \in V_{i+1}^C$ such that $i(w) = u$, because $C$ is $i$-saturated. Similarly, there exist an edge $e \in E_{i+1}$ and a vertex $w' \in V_{i+1}^C$ such that $s(e) = u, t(e) = w'$, because $C$ is $\lambda$-saturated. Let $u, v$ be vertices with $u \in V_i^C, v \in V_{i+2}$, and $w \geq v' = i(u)$. As $C$ is $i$-hereditary, we have that $v$ belongs to $V_{i+1}^C$. As $C$ is $\lambda$-hereditary, if an edge $e \in E_{i+1}$ satisfies $t(e) = v$, one sees that $s(e)$ belongs to $V_{i+1}^C$ and hence $e$ belongs to $E_{i+1}^C$. Therefore $(V'^C, E'^C, \lambda'^C, i'^C)$ inherits the local property of $\Sigma$. Thus $(V'^C, E'^C, \lambda'^C, i'^C)$ becomes a $\lambda$-graph system. \qed
We denote by $\mathcal{L}^{\lambda C}$ the $\lambda$-graph system $(V^{\lambda C}, E^{\lambda C}, \lambda^{\lambda C}, \iota^{\lambda C})$ and call it the $\lambda$-graph subsystem of $\mathcal{L}$ removed by removing $C$. Let $\mathcal{I}_C$ be the closed ideal of $O_C$ generated by the projections $E^i_j$ for $v^i_j \in C$, that is, $\mathcal{I}_C = O_C(E^i_j \mid v^i_j \in C)O_C$ the closure of $O_C\{E^i_j \mid v^i_j \in C\}O_C$.

**Lemma 3.2.** The set of all linear combinations of elements of the form

$$S_{i}E^m_jS^*_n, \quad \text{for } v^i_j \in C, \mu, \nu \in \Lambda^*$$

is dense in $\mathcal{I}_C$.

**Proof.** Since the finite linear combinations of elements of the form $S_i E^m_j S^*_n$ for $|\xi|, |\eta| \leq p, f = 1, \ldots, m(p)$ is dense in $O_C$, elements of the form

$$S_i E^m_j S^*_n, \quad \text{for } v^i_j \in C, |\xi|, |\eta| \leq p, |\xi|, |\eta| \leq q$$

span the ideal $\mathcal{I}_C$. Put $T = S_i E^m_j S^*_n E^p_j S^*_p E^p_j S^*_p$ and assume $T \neq 0$. The equality

$$S^*_n E^m_j S^*_n = \sum_{j=1}^{m(p)l+i} A_{l,j+i} (i, \eta, j) E^l_{i+j+\eta}$$

holds, where $A_{l,j+i}(i, \eta, j) = 1$, if there exists a $\lambda$-path from $v^i_j$ to $v^j_{i+\eta}$ with label $\eta$, otherwise $A_{l,j+i}(i, \eta, j) = 0$. The vertex $v^i_j$ belongs to $C$ if $A_{l,j+i}(i, \eta, j) = 1$, because $v^i_j \in C$ and $C$ is $\lambda$-hereditary. As $T = S_i E^m_j S^*_n E^p_j S^*_p E^p_j S^*_p$ and we may assume that $l$ is large enough, $T$ is assumed to be of the form $T = S_i E^m_j S^*_n E^p_j S^*_p$ for $v^i_j \in C$. As $T \neq 0$, the element $E^m_j S^*_n S^*_n$ is either of the form $E^m_j S^*_n$, or $E^m_j S^*_n$ for some word $v$. In the former case, we have $T = S_i S^*_n E^m_j S^*_n$. Since $S^*_n E^m_j S^*_n$ is a finite linear combination of $E^m_j S^*_n$ for $v^i_j \in C$ and $l$ is large enough, $T$ is a finite linear combination of elements of the form (3.1), because $C$ is $\lambda$-hereditary. In the latter case, we have $T = S_i S^*_n E^m_j S^*_n S^*_p S^*_p$. Since $S^*_n E^m_j S^*_n$ is a finite linear combination of $E^m_j S^*_n$ for $v^i_j \in C$ and $l$ is large enough, we have $T = S_i E^m_j S^*_n$. Hence we get the desired assertion.

**Lemma 3.3.** If $E^i_j$ belongs to the ideal $\mathcal{I}_C$, the vertex $v^i_j$ belongs to the set $C$.

**Proof.** For $k \leq l$, set

$$E_{k,l} = \sum_{|\mu|=k, |\nu|=l} S_{\mu}E^\mu_j S^*_\nu$$

belonging to $\mathcal{I}_C$. For an operator $T = S_i E^m_j S^*_n$ with $v^i_j \in C$, it follows that $TE_{k,l} = E^m_j T = T$ for large enough $k, l$. Lemma 3.2 says that $\{E_{k,l}\}_{k,l}$ is an approximate unit.
for \( T_\mathcal{I} \). Suppose that a vertex \( v_l^j \in V \) does not belong to \( C \). It suffices to show that the equality

\[
\| E^l_k E_{l,l} - E^l_k \| = 1
\]

holds for all large enough \( k, l \). We fix \( k \leq l \) large enough. We may assume that \( E^l_k E_{l,l} \neq 0 \) and \( L + k \leq l \). There exists an admissible word \( \mu \) of length \( k \) such that \( S^*_\mu E^l_k S_\mu E^l_k \neq 0 \) and hence \( S^*_\mu E^l_k S_\mu \geq E^l_k \). On the other hand, \( C \) is saturated, so we may find a \( \lambda \)-path \( \pi \) in \( E_{i,l+l} \) whose source vertex \( s(\pi) \) is \( v_l^j \), and an \( c \)-path from the terminal vertex \( t(\pi) \) of \( \pi \) to a vertex \( v_p^j \) that does not belong to \( C \). We set \( \gamma = \lambda(\pi) \) the label of \( \pi \) so that \( S^*_\gamma E^l_k S_\gamma \geq E^l_k \). It then follows that

\[
E^l_k \geq S_\gamma S^*_\gamma E^l_k S_\gamma + S_\gamma S^*_\gamma E^l_k S_\gamma S_\gamma E^l_k S^*_\gamma \geq S_\gamma E^l_k S^*_\gamma + S_\gamma E^l_k S^*_\gamma.
\]

Since \( \sum_{|l|=k,v^j \in \cal{C}} S_l E^l_k S^*_l \) is orthogonal to \( S_\gamma E^l_k S^*_\gamma \), one obtains that

\[
E^l_k E_{l,l} - E^l_k \geq S_\gamma E^l_k S^*_\gamma
\]

so that (3.2) holds.

**Lemma 3.4.** For any nonzero closed ideal \( \mathcal{I} \) of the \( C^* \)-algebra \( \mathcal{O}_\Sigma \), put

\[
C_\mathcal{I} = \{ v^j_l \in V \mid E^l_k \in \mathcal{I} \}.
\]

Then \( C_\mathcal{I} \) is a nonempty saturated hereditary subset of \( V \).

**Proof.** Since \( \Sigma \) satisfies condition (I), the set \( C_\mathcal{I} \) is nonempty because of the uniqueness of the algebra \( \mathcal{O}_\Sigma \). Take \( v^j_l \in C_\mathcal{I} \). Suppose that \( v^{j+1} \) satisfies \( v^j_l \geq v^{j+1}_l \). The inequality \( E^l_k \geq E^{l+1}_k \) assures \( E^{l+1}_k \in \mathcal{I} \). Suppose next \( v^j_l \geq v^{j+1}_l \). There exists a symbol \( \alpha \in \Sigma \) such that \( A_{i,j+1}(i, \alpha, j) = 1 \). By (2.4), we have \( S^*_\alpha E^l_k S_\alpha \geq E^{l+1}_k \) so that \( E^{l+1}_k \in \mathcal{I} \). Hence \( C_\mathcal{I} \) is hereditary. For \( \delta \), suppose that \( v^j_l \geq v^{j+1}_l \) implies \( v^{j+1}_l \in C_\mathcal{I} \). This means that \( A_{i,j+1}(i, \alpha, j) = 1 \) implies \( E^{l+1}_k \in \mathcal{I} \). By the second equality of (2.2), we see \( E^l_k \in \mathcal{I} \). Suppose next that \( v^j_l \geq v^{j+1}_l \) implies \( v^{j+1}_l \in C_\mathcal{I} \). This means that \( A_{i,j+1}(i, \alpha, j) = 1 \) implies \( E^{l+1}_k \in \mathcal{I} \). By (2.4), we have \( S^*_\alpha E^l_k S_\alpha \in \mathcal{I} \) for all \( \alpha \in \Sigma \), so that \( E^l_k = \sum_{\alpha \in \Sigma} S^*_\alpha E^l_k S_\alpha \) belongs to \( \mathcal{I} \). Thus \( \mathcal{I} \) is saturated.

**Proposition 3.5.** Let \( \Sigma = (V, E, \lambda, t) \) be a \( \lambda \)-graph system satisfying condition (I).

Let \( C \) be a saturated hereditary subset of \( V \). A vertex \( v^j_l \) belongs to \( C \) if and only if \( E^l_k \) belongs to \( T_\mathcal{I} \). Hence there exists a bijective correspondence between the set of all saturated hereditary subsets of \( V \) and the set of all ideals in \( \mathcal{O}_\Sigma \).

**Proof.** Let \( C \) be a saturated hereditary subset of \( V \). For a vertex \( v^j_l \in V \), we have \( v^j_l \in C \) if and only if \( E^l_k \in \mathcal{I} \) by Lemma 3.3. For an ideal \( \mathcal{I} \) of \( \mathcal{O}_\Sigma \), we have \( E^l_k \in \mathcal{I} \) if and only if \( v^j_l \in C_\mathcal{I} \) by definition of \( C_\mathcal{I} \). Hence we conclude the assertions.
DEFINITION. A $\lambda$-graph system $\mathcal{L}$ satisfies condition (II) if for every saturated hereditary subset $C \subset V$, the $\lambda$-graph system $\mathcal{L}^C$ satisfies condition (I).

Let $A$ be an $n \times n$ square matrix with entries in $\{0, 1\}$. Then $A$ satisfies condition (II) in the sense of Cuntz [6] if and only if the natural $\lambda$-graph system $O^A_\mathcal{L}$ constructed from $A$ satisfies condition (II) in the above sense (compare Section 5).

THEOREM 3.6. Suppose that a $\lambda$-graph system $\mathcal{L}$ satisfies condition (II). For an ideal $I$ of $O^A_\mathcal{L}$, the quotient $C^*$-algebra $O^A_\mathcal{L}/I$ is isomorphic to the $C^*$-algebra $O^A_\mathcal{L}/C^I$ associated with the $\lambda$-graph system $\mathcal{L}^C$ obtained from $\mathcal{L}$ by removing the saturated hereditary subset $C_I$ for $I$.

PROOF. We denote by $S_{\alpha}, E_{\alpha}'$ the quotient images of $S_{\alpha}, E_{\alpha}'$ in the quotient $C^*$-algebra $O^A_\mathcal{L}/I$ respectively. Let $s_{\alpha}, e_{\alpha}'$ be the canonical generating partial isometries for $\alpha \in \Sigma$ and the projections corresponding to the vertices $v_{\alpha}'$ of $V^C$ in $O^A_\mathcal{L}/C^I$. Since we have $E_{\alpha}' \neq 0$ if and only if $v_{\alpha}' \in V^C$, the relations
\[
\overline{S_{\alpha}' E_{\alpha}'} = \sum_{k=1}^{m(I+1)} A_{\alpha,i,k} E_{\alpha}' E_{\alpha}''', \quad \text{for $\alpha \in \Sigma$}
\]
hold. By the uniqueness of the algebras $O^A_\mathcal{L}$ and $O^A_\mathcal{L}/I$, subject to the operator relations, the correspondence $S_{\alpha} \leftrightarrow s_{\alpha}, E_{\alpha}' \leftrightarrow e_{\alpha}'$ for $\alpha \in \Sigma$, $v_{\alpha}' \in V^C$ extends to an isomorphism between $O^A_\mathcal{L}/I$ and $O^A_\mathcal{L}/C^I$.

COROLLARY 3.7. In the $\lambda$-graph systems satisfying condition (II), the class of the $C^*$-algebras associated with $\lambda$-graph systems is closed under quotients by its ideals.

We say a closed ideal $J$ of $A_\mathcal{L}$ to be saturated if $\lambda_{\mathcal{L}}(E_{\alpha}') \in J$ implies $E_{\alpha}' \in J$. We are assuming that a $\lambda$-graph system $\mathcal{L}$ satisfies condition (I).

LEMMMA 3.8. For an ideal $I$ of $A_\mathcal{L}$, set $J = I \cap A_\mathcal{L}$. Then $J$ is a nonzero $\lambda_\mathcal{L}$-invariant saturated ideal of $A_\mathcal{L}$.

PROOF. It suffices to show that $J$ is saturated. Suppose that $\lambda_{\mathcal{L}}(E_{\alpha}') \in J$. We see $S_{\alpha}' E_{\alpha} S_{\alpha}$ belongs to $J$ for each $\alpha \in \Sigma$. Hence $E_{\alpha}' = \sum_{\alpha \in \Sigma} S_{\alpha}' E_{\alpha} S_{\alpha}'$ belongs to $J$.

LEMMMA 3.9. There exists a bijective correspondence between the set of $\lambda_{\mathcal{L}}$-invariant closed saturated ideals of $A_\mathcal{L}$ and the set of saturated hereditary subsets of $V$.

PROOF. Let $J$ be a $\lambda_{\mathcal{L}}$-invariant saturated ideal of $A_\mathcal{L}$. Put $C_J = \{ v_{\alpha}' \in V \mid E_{\alpha}' \in J \}$. As $J$ is $\lambda_{\mathcal{L}}$-invariant, we have $\sum_{\alpha \in \Sigma} S_{\alpha}' E_{\alpha} S_{\alpha}$ belongs to $J$ for $v_{\alpha}' \in C_J$. Hence
Theorem 3.10. Consider the following six conditions.

(i) $\mathcal{O}_E$ is simple.
(ii) There is no nontrivial $\lambda_E$-invariant saturated ideal of $\mathcal{A}_E$.
(iii) There is no proper saturated hereditary subset of $V$.
(iv) $\mathcal{L}$ is irreducible.
(v) There is no nontrivial $\lambda_E$-invariant ideal of $\mathcal{A}_E$.
(vi) There is no proper hereditary and $\iota$-saturated subset of $V$.

Conditions (i)–(iii) are equivalent to each other, and also conditions (iv)–(vi) are equivalent to each other. The latter conditions imply the former conditions.

Proof. As nontrivial ideals of $\mathcal{O}_E$ bijectively correspond to saturated hereditary subsets of $V$, the first three conditions are equivalent each other. It suffices to show that (iv) is equivalent to (vi). Assume that $\mathcal{L}$ is irreducible. Let $C$ be a nonempty hereditary and $\iota$-saturated subset of $V$. Take a vertex $v_i^l \in C$. Let $U_N(v_i^l)$ be the set of $\iota$-orbits $u = (u_n)_{n \in \mathbb{Z}_+} \in \Omega_E$ such that there exists a $\lambda$-path of length $N$ from $v_i^l$ to the vertex $u_{i+N}$. Since $\mathcal{L}$ is irreducible, we have $\Omega_E = \bigcup_{N=0}^{\infty} U_N(v_i^l)$. Hence there exist $N_1, N_2, \ldots, N_k$ such that $\Omega_E = \bigcup_{j=1}^{k} U_N(v_i^l)$, because $U_N(v_i^l)$ is open in $\Omega_E$. We may assume that $0 \leq N_1 \leq N_2 \leq \cdots \leq N_k$. We put $N_k = L$. For a vertex $w \in V_{i+L}$, find an $\iota$-orbit $x = (x_n)_{n \in \mathbb{Z}_+} \in \Omega_E$ such that $x_{i+L} = w$. Take $N_k$ such that $x \in U_{N_k}(v_i^l)$. Since $C$ is $\lambda$-hereditary and $\iota$-saturated, we see $x_{i+N_k} \in C$ and hence $w \in C$. This implies $V_{i+N_k} \subset C$. Now $C$ is $\iota$-saturated, so we conclude that $V = C$. Therefore we get the implication from (iv) to (vi).

Suppose that $\mathcal{L}$ is not irreducible. There exists an $\iota$-orbit $u = (u_n)_{n \in \mathbb{Z}_+} \in \Omega_E$ and a vertex $v_i^l$ such that $u$ does not belong to $\bigcup_{N=0}^{\infty} U_N(v_i^l)$. Let $V^N(v_i^l)$ be the set of all vertices $w$ in $V_{i+N}$ that are terminal vertices of $\lambda$-edges whose source vertices are $v_i^l$. Put $V(v_i^l) = \bigcup_{N=0}^{\infty} V^N(v_i^l)$ and $W(v_i^l) = \{ w \in V \mid v_i^l \leq w \text{ for some vertex } v \in V(v_i^l) \} \cup V(v_i^l)$. 


By the local property of the $\lambda$-graph system, the set $W(v_j^i)$ is $\lambda$-hereditary and the vertices $u_n$ do not belong to $W(v_j^i)$ for all $n \in \mathbb{Z}_+$. It is by definition that $W(v_j^i)$ is $\iota$-hereditary. Let $C$ be the saturation of $W(v_j^i)$ with respect to $\geq$. As $W(v_j^i)$ is $\lambda$-hereditary, $C$ is so from the local property of $\lambda$-graph system. It is obvious that $C$ is $\iota$-hereditary. We obtain a proper hereditary and $\iota$-saturated subset $C$ of $V$. □

4. Structure of ideals

In this section, we prove that an ideal of $O_\mathcal{C}$ is stably isomorphic to the $C^*$-subalgebra of $O_\mathcal{C}$ associated with the corresponding saturated hereditary subset of $V$. As a result, we can present the K-theory formulae for ideals of $O_\mathcal{C}$ in terms of the corresponding saturated hereditary subsets of $V$. The notation is as in the previous sections. For a saturated hereditary subset $C$ of $V$, put for $v_j^i \in C$

$$\Lambda^C(v_j^i) = \left\{ \mu \in \Lambda^* \left| \begin{array}{l}
\text{there exists a } \lambda\text{-path } \pi \text{ such that } \lambda(\pi) = \mu.
\end{array} \right. \right\},$$

where $s(\pi)$ and $t(\pi)$ are the source vertex and the terminal vertex of $\pi$ respectively. We denote by $O_\mathcal{C}(C)$ the $C^*$-subalgebra of $O_\mathcal{C}$ generated by elements of the form $S_{\mu}E_j^iS_\nu^j$, for $\mu, \nu \in \Lambda^C(v_j^i)$, $v_j^i \in C$.

**Lemma 4.1.** The set of all finite linear combinations of elements of the form $S_{\mu}E_j^iS_\nu^j$, for $\mu, \nu \in \Lambda^C(v_j^i)$, $v_j^i \in C$, is a dense $*$-subalgebra of $O_\mathcal{C}(C)$.

**Proof.** For $v_j^i, v_j^k \in C$, $\mu, \nu \in \Lambda^C(v_j^i)$, $\xi, \eta \in \Lambda^C(v_j^k)$, suppose that

$$S_{\mu}E_j^iS_\nu^jS_t^iE_j^kS_\eta^j \neq 0.$$

We may assume $|\nu| > |\xi|$. We then have $\nu = \xi\nu'$ for some $\nu'$, so that

$$S_{\mu}E_j^iS_\nu^jS_t^iE_j^kS_\eta^j = S_{\mu}E_j^iS_\xi^jS_t^iE_j^kS_\eta^j.$$

If $|\nu'| + k < l$, we have that $E_j^iS_\xi^jS_t^iE_j^kS_\eta^j = E_j^l$. If $|\nu'| + k \geq l$, we see that $E_j^iS_\xi^jS_t^iE_j^kS_\eta^j$ is a finite sum of projections $E_j^{\nu'|k}$ with $v_j^{\nu'|k} \in C$. In both cases, $S_{\mu}E_j^iS_\xi^jS_t^iE_j^kS_\eta^j$ is a finite linear combination of $S_\xi^jE_\delta^mS_\eta^j$ with $\xi, \delta \in \Lambda^C(v_j^m)$, $v_j^m \in C$. □

We prove that the ideal $I_C$ of $O_\mathcal{C}$ is stably isomorphic to the $C^*$-algebra $O_\mathcal{C}(C)$ under some condition. Put $P_l = \sum_{i, v_j^i \in C} E_j^i$ for $l \in \mathbb{N}$. It belongs to the algebra $O_\mathcal{C}(C)$ and satisfies $P_l \leq P_{l+1}$. We see then a sequence of natural embeddings $P_lO_\mathcal{C}P_l \subset P_{l+1}O_\mathcal{C}P_{l+1} \subset \cdots$.

**Proposition 4.2.** $O_\mathcal{C}(C) = \lim_{l \to \infty} P_lO_\mathcal{C}P_l$. 
**PROOF.** We first prove the inclusion relation $\mathcal{O}_C(C) \subset \lim_{n \to \infty} P_l \mathcal{O}_C P_l$. For $v^l_i \in C$ and $\mu \in \Lambda^C(v^l_i)$, take a $\lambda$-path $\pi$ such that $s(\pi) \in C$, $t(\pi) = v^l_i$, and $\lambda(\pi) = \mu$. We put $s(\pi) = v^l_i$. The projection $E^l_i$ satisfies the inequality $S^l_i E^l_i S^l_i \geq E^l_i$ so that $E^l_i S^l_i E^l_i = S^l_i E^l_i$. As $\mathcal{L}$ is left-resolving, we know that $S^l_i E^l_i S^l_i = 0$ for $k_1 \neq j_1$. It then follows that $P_l S^l_i E^l_i = S^l_i E^l_i$. Symmetrically we have that $E^l_i S^l_l P_l = E^l_l S^l_l$ for some $l_2$. Hence we see that $P_l S^l_i E^l_i S^l_l P_l = S^l_i E^l_i S^l_l$. Thus we have proved that for $v^l_i \in C$ and $\mu, v \in \Lambda^C(v^l_i)$, there exists $M \in \mathbb{N}$ such that $P_m S^l_i E^l_i S^l_l = S^l_i E^l_l S^l_m$ for all $m \geq M$. This implies the inclusion relation $\mathcal{O}_C(C) \subset \lim_{n \to \infty} P_l \mathcal{O}_C P_l$.

For $v^l_i \in V$, $\mu, v \in \Lambda^*$, and $v^l_i, v^l_i \in C$, we next prove that the element $E^l_i S^l_j E^l_i S^l_j E^l_i$ belongs to the algebra $\mathcal{O}_C(C)$. We may assume that $l$ is large enough because of the second relation of (2.2). Assume $S^l_i E^l_i S^l_j E^l_i S^l_j \neq 0$ so that $S^l_i E^l_i S^l_j \geq E^l_i$. Hence there exists a $\lambda$-path whose source is $v^l_i$ and terminal is connected to $v^l_i$ by an $\iota$-path. By the local property of the $\lambda$-graph system, we may find a $\lambda$-path $\pi$ in $E$ such that $\lambda(\pi) = \mu$, $t(\pi) = v^l_i$ and an $\iota$-path that connects between $s(\pi)$ and $v^l_i$. Since $v^l_i$ belongs to $C$ and $C$ is hereditary, we see that $v^l_i \in C$ and $\mu$ belongs to $\Lambda^C(v^l_i)$. Symmetrically one sees that $v$ belongs to $\Lambda^C(v^l_i)$ from the inequality $S^l_i E^l_i S^l_j \geq E^l_i$. Hence we have $E^l_i S^l_i E^l_i S^l_j E^l_i = S^l_i E^l_i S^l_j$ and it belongs to the algebra $\mathcal{O}_C(C)$. Thus we have $\lim_{n \to \infty} P_l \mathcal{O}_C P_l \subset \mathcal{O}_C(C)$. □

**Theorem 4.3.** The ideal $\mathcal{I}_C$ is stably isomorphic to the algebra $\mathcal{O}_C(C)$.

**PROOF.** Let $X_l = \mathcal{O}_C P_l$ for $l \in \mathbb{N}$. Then $X_l$ has a Hilbert left $\mathcal{O}_C P_l \mathcal{O}_C$-module and a Hilbert right $\mathcal{O}_C P_l \mathcal{O}_C$-module structure in a natural way. Its left $\mathcal{O}_C P_l \mathcal{O}_C$-valued inner product and right $\mathcal{O}_C P_l \mathcal{O}_C$-valued inner product are given by

$$\langle a P_l, b P_l \rangle_L = a P_l b^*, \quad \langle a P_l, b P_l \rangle_R = P_l a^* b P_l,$$

for $a, b \in \mathcal{O}_C$ respectively. Hence the norms on $X_l$ coming from their respect inner products coincide with the norm on the $C^*$-algebra $\mathcal{O}_C$. As $P_l \leq P_{l+1}$, we have a natural embedding $X_l \hookrightarrow X_{l+1}$. Let $X_C$ be the closure of $\bigcup_{l = 1}^{\infty} X_l$ in the norm of $\mathcal{O}_C$, that is regarded as the inductive limit of the inclusions $X_l \hookrightarrow X_{l+1}$, $l \in \mathbb{N}$. The ideal $\mathcal{I}_C$ and the algebra $\mathcal{O}_C(C)$ are the inductive limits $\lim_{\to \infty} \mathcal{O}_C P_l \mathcal{O}_C$ and $\lim_{\to \infty} \mathcal{O}_C P_l \mathcal{O}_C$ respectively. We then see that the subspace $X_C$ of $\mathcal{O}_C$ has an induced left $\mathcal{I}_C$-valued inner product and right $\mathcal{O}_C(C)$-valued inner product such as

$$\langle \xi, \eta \rangle_L = \xi^* \eta \in \mathcal{I}_C, \quad \langle \xi, \eta \rangle_R = \xi^* \eta \in \mathcal{O}_C(C),$$

for $\xi, \eta \in X_C$ respectively. It also has a natural left $\mathcal{I}_C$-module and right $\mathcal{O}_C(C)$-module structures respectively. It is easy to see that both the linear spans of $\langle \xi, \eta \rangle_L$, for $\xi, \eta \in X_C$, and $\langle \xi, \eta \rangle_R$, for $\xi, \eta \in X_C$, are dense in $\mathcal{I}_C$ and $\mathcal{O}_C(C)$ respectively. Hence $X_C$ is a full Hilbert left $\mathcal{I}_C$-module, and a full Hilbert right $\mathcal{O}_C(C)$-module such
that \((\xi, \eta)_{\mathbb{R}} = \xi(\eta, \zeta)_{\mathbb{R}}\) for \(\xi, \eta, \zeta \in \mathcal{X}_C\). This means that \(\mathcal{X}_C\) is an \(\mathcal{I}_C\) - \(\mathcal{O}_\mathbb{R}(C)\) imprimitivity bimodule, so that \(\mathcal{I}_C\) and \(\mathcal{O}_\mathbb{R}(C)\) are Morita equivalent ([32]). By [4], they are stably isomorphic to each other.

By using the above result, we next compute the K-theory of the ideal \(\mathcal{I}_C\). The subalgebra \(\mathcal{O}_\mathbb{R}(C)\) is invariant globally under the gauge action \(\alpha_\mathbb{R}\) on \(\mathcal{O}_\mathbb{R}\). We still denote by \(\alpha_\mathbb{R}\) the restriction of \(\alpha_\mathbb{R}\) to \(\mathcal{O}_\mathbb{R}(C)\). We denote by \(\mathcal{F}_\mathbb{R}(C)\) the \(\mathcal{C}^*\)-subalgebra of \(\mathcal{O}_\mathbb{R}(C)\) generated by \(S_{\mu}E_i^jS_{\mu}^*\), \(\mu, v \in \Lambda^\mathbb{C}(v'_i), |\mu| = |v|, v'_i \in \mathbb{C}_+\). That is, \(\mathcal{F}_\mathbb{R}(C) = \mathcal{F}_\mathbb{R} \cap \mathcal{I}_C\). It is direct to see that the fixed point algebra \(\mathcal{O}_\mathbb{R}(C)^{\alpha_\mathbb{R}}\) of \(\mathcal{O}_\mathbb{R}(C)\) under \(\alpha_\mathbb{R}\) is the algebra \(\mathcal{F}_\mathbb{R}(C)\). A similar discussion to [22] (compare [24]) assures that the crossed product \(\mathcal{O}_\mathbb{R}(C) \rtimes_{\alpha_\mathbb{R}} \mathbb{T}\) is stably isomorphic to \(\mathcal{F}_\mathbb{R}(C)\). We can show the following result.

**Lemma 4.4** (compare [24, Lemma 7.5], [22, Lemma 4.3]).

(i) \(K_0(\mathcal{O}_\mathbb{R}(C)) \cong K_0(\mathcal{O}_\mathbb{R}(C) \rtimes_{\alpha_\mathbb{R}} \mathbb{T})/(id - \alpha_\mathbb{R}^{-1})K_0(\mathcal{O}_\mathbb{R}(C) \rtimes_{\alpha_\mathbb{R}} \mathbb{T})\).

(ii) \(K_1(\mathcal{O}_\mathbb{R}(C)) \cong \text{Ker}(id - \alpha_\mathbb{R}^{-1})\text{ on } K_0(\mathcal{O}_\mathbb{R}(C) \rtimes_{\alpha_\mathbb{R}} \mathbb{T})\),
where \(\alpha_\mathbb{R}\) is the dual action of \(\alpha_\mathbb{R}\).

Let \(\mathcal{F}_k(C)\) be the \(\mathcal{C}^*\)-subalgebra of \(\mathcal{F}_\mathbb{R}(C)\) generated by \(S_{\mu}E_i^jS_{\mu}^*, \mu, v \in \Lambda^\mathbb{C}(v'_i), |\mu| = |v| = k, v'_i \in \mathcal{C} \cap \mathcal{V}_l\) and \(\mathcal{F}_k^\infty(C)\) the \(\mathcal{C}^*\)-subalgebra of \(\mathcal{F}_\mathbb{R}(C)\) generated by \(\mathcal{F}_k^l(C), k \leq l \in \mathbb{N}\). Hence we see that

\[\mathcal{F}_k^l(C) = \mathcal{F}_k \cap \mathcal{O}_\mathbb{R}(C), \quad \mathcal{F}_k^\infty(C) = \mathcal{F}_k^\infty \cap \mathcal{O}_\mathbb{R}(C)\]

The embeddings of \(\iota_{l,l+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1}\) and \(\lambda_{k,k+1} : \mathcal{F}_k^\infty \hookrightarrow \mathcal{F}_{k+1}^\infty\) of the original AF-algebra \(\mathcal{F}_\mathbb{R}\), are inherited in the algebras \(\mathcal{F}_k^l(C), \mathcal{F}_k^\infty(C), \mathcal{F}_\mathbb{R}(C)\), so that \(\mathcal{F}_\mathbb{R}(C)\) is an AF-algebra. Let \(m_\mathbb{C}(l)\) be the cardinal number of the vertex set \(\mathcal{C} \cap \mathcal{V}_l\). We put \(\mathcal{C} \cap \mathcal{V}_l = \{u_l^j, u_l^{j+1}, \ldots, u_{m_\mathbb{C}(l)}^j\}\). Define the following matrices:

\[A(C)_{l,l+1}(i, j, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = u_l^j, t(e) = \alpha, \text{ for some } e \in E_{l,l+1}^i, \\ 0 & \text{otherwise,} \end{cases}\]

\[I(C)_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(u_l^{j+1}) = u_l^j, \\ 0 & \text{otherwise,} \end{cases}\]

\[A(C)_{l,l+1}(i, j) = \sum_{\alpha \in \Sigma} A(C)_{l,l+1}(i, \alpha, j),\]

for \(i = 1, 2, \ldots, m_\mathbb{C}(l), j = 1, 2, \ldots, m_\mathbb{C}(l+1)\). Let

\[D(C)_{l,l+1} = I(C)_{l,l+1} - A(C)_{l,l+1} : \mathbb{Z}^{m_\mathbb{C}(l)} \to \mathbb{Z}^{m_\mathbb{C}(l+1)}, \quad l \in \mathbb{Z}_+.\]
As \( I(C)_{l+1,l+2} A(C)_{l+1} = A(C)_{l+1,l+2} I(C)_{l+1} \), the matrix \( I(C)_{l+1,l+2} \) induces a homomorphism from \( \mathbb{Z}^{m(l+1)} / D(C)_{l+1,l+2} \mathbb{Z}^{m(l)} \) to \( \mathbb{Z}^{m(l+2)} / D(C)_{l+1,l+2} \mathbb{Z}^{m(l+1)} \) that is denoted by \( T(C)_{l+1,l+2} \). Thanks to Theorem 4.3, we can present the K-theory formulae for ideals of \( \mathcal{O}_\Sigma \).

**Theorem 4.5.** Let \( \Sigma \) be a \( \lambda \)-graph system satisfying condition (II). Let \( \mathcal{I} \) be an ideal of \( \mathcal{O}_\Sigma \) and \( C \) its corresponding saturated hereditary subset of the vertex set of \( \Sigma \).

Then we have

\[
K_0(\mathcal{I}) \cong \lim_{\rightarrow} \left\{ \mathbb{Z}^{m(l+1)} / D(C)_{l+1,l+2} \mathbb{Z}^{m(l)} ; T(C)_{l+1,l+2} \right\},
\]

\[
K_1(\mathcal{I}) \cong \lim_{\rightarrow} \left\{ \text{Ker} D(C)_{l+1,l+1} \text{ in } \mathbb{Z}^{m(l)} ; I(C)_{l+1,l+1} \right\}.
\]

Although the \( C^* \)-algebra \( \mathcal{O}_\Sigma \) is not necessarily defined by a \( \lambda \)-graph system, in the case when \( C \) has a bounded upper bound, it is given by a \( \lambda \)-graph system. Let

\[ V_i^C = C \cup \{ v \in V \mid \text{there exists } u_0 \in C \text{ such that } t^m(u_0) = v \text{ for some } m \in \mathbb{N} \}. \]

A saturated hereditary subset \( C \) of \( V \) is said to have a bounded upper bound if the cardinality of the set \( V_i^C \setminus C \) is finite. It is equivalent to the condition that there exists \( L \in \mathbb{N} \) such that \( V_n \cap V_i^C = V_n \cap C \) for all \( n \geq L \). We will assume that \( C \) has a bounded upper bound. Take \( L \in \mathbb{N} \) as above. Define for \( l \in \mathbb{Z}_+ \)

\[
V_i^C = C \cap V_{i,L}, \\
E_{i,j+i}^C = \left\{ e \in E_{i+j+i,L+j+i} \mid s(e) \in V_i^C, t(e) \in V_j^C \right\}, \\
\lambda^C = \lambda|_{E^C}, \\
i_{i,j+i}^C = \iota|_{E^C}.
\]

Since \( V_i^C \cap V_{i+L} = C \cap V_{i,L} \), one sees that \( t(u) \in V_i^C \) for \( u \in V_i^C \). It is straightforward to see that \( (V_i^C, E_{i,j+i}^C, \lambda^C, i_{i,j+i}^C)_{i \in \mathbb{Z}_+} \) yields a \( \lambda \)-graph system, denoted by \( \Sigma^C \). We note that \( C \) has a bounded upper bound if and only if there exists \( L \in \mathbb{N} \) such that \( P_l = P_{L} \) for all \( l \geq L \).

**Proposition 4.6.** Let \( \Sigma \) be a \( \lambda \)-graph system satisfying condition (II). If a saturated hereditary subset \( C \) of \( V \) has a bounded upper bound, the algebra \( \mathcal{O}_\Sigma(C) \) is isomorphic to the \( C^* \)-algebra \( \mathcal{O}_{\Sigma^C} \) associated with the \( \lambda \)-graph system \( \Sigma^C \). Hence the ideal \( \mathcal{I}^C \) is stably isomorphic to the \( C^* \)-algebra \( \mathcal{O}_{\Sigma^C} \).

**Proof.** Take \( L \in \mathbb{N} \) such that \( V_n \cap V_i^C = V_n \cap C \) for all \( n \geq L \). As \( P_l = P_{L} \) for all \( l \geq L \), one has \( \mathcal{O}_\Sigma(C) = P_{L} \mathcal{O}_\Sigma P_{L} \) by Proposition 4.2. Let \( \Sigma^{(L)} = (V^{(L)}, E^{(L)}, \lambda^{(L)}, \iota^{(L)}) \) be the \( L \)-shift \( \lambda \)-graph system of \( \Sigma \) defined by

\[
V_i^{(L)} = V_{i,L}, \\
E_{i,j+i}^{(L)} = E_{i+j+i,L+j+i}, \\
\lambda^{(L)} = \lambda|_{E^{(L)}}, \\
i_{i,j+i}^{(L)} = \iota|_{E^{(L)}}.
\]
for $l \in \mathbb{Z}_+$. By [28, Proposition 2.3], the algebra $\mathcal{O}_{\Sigma}$ coincides with the algebra $\mathcal{O}_{\Sigma^{(2)}}$. It is direct to see that $P_L\mathcal{O}_{\Sigma^{(2)}}P_L$ is isomorphic to $\mathcal{O}_{\Sigma}$. Hence $\mathcal{O}_{\Sigma}(C)$ is isomorphic to $\mathcal{O}_{\Sigma}$.

5. Examples

Let $G = (V, E)$ be a finite directed graph with finite vertex set $V$ and finite edge set $E$. Let $\mathcal{G} = (G, \lambda)$ be a labeled graph over an alphabet $\Sigma$ defined by $G$ and a labeling map $\lambda : E \to \Sigma$. Suppose that it is left-resolving and predecessor-separated (see [19]). Let $A_G$ be the adjacency matrix of $G$ that is defined by

$$A_G(e, f) = \begin{cases} 1 & \text{if } t(e) = s(e), \\ 0 & \text{otherwise,} \end{cases}$$

for $e, f \in E$. The matrix $A_G$ defines a shift of finite type by regarding the edge set $E$ as its alphabet. Since the matrix $A_G$ has entries in $\{0, 1\}$, we have the Cuntz-Krieger algebra $\mathcal{O}_{A_G}$ defined by $A_G$ ([7] compare [18, 33]). By putting $V^{G}_{l+1} = V$, $E^{G}_{l+1} = E$ for $l \in \mathbb{Z}_+$, and $\lambda^G = \lambda$, $\epsilon^G = \text{id}$, we have a $\lambda$-graph system $\mathcal{L}_G = (V^G, E^G, \lambda^G, \epsilon^G)$. The $C^*$-algebra $\mathcal{O}_{\Sigma^{G}}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{A_G}$ ([24, Proposition 7.1]).

Let us consider the following labeled graph. The vertex set $V$ is $\{v_1, v_2, v_3\}$. The edges labeled $\alpha$ are from $v_2$ to $v_3$ and from $v_3$ to $v_2$ and a self-loop at $v_1$. The edges labeled $\beta$ are self-loops at $v_1$ and at $v_3$. The edge labeled $\gamma$ is from $v_1$ to $v_2$. The resulting labeled graph is denoted by $\mathcal{G}$. The $\lambda$-graph system $\mathcal{L}_G$ is left-resolving and satisfies condition (II). In $\mathcal{L}_G$, let $C$ be the vertex set corresponding to $\{v_2, v_3\}$. It is saturated hereditary. The $\lambda$-graph subsystem $\mathcal{L}_G^C$ of $\mathcal{L}_G$ obtained by removing $C$ consists of one $\iota$-orbit of the vertex $\{v_1\}$ with two self-loops labeled $\alpha$ and $\beta$. Hence we have

$$\mathcal{O}_{\Sigma^G} \cong \mathcal{O}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \quad \mathcal{O}_{\Sigma^G}/\mathcal{I}_C \cong \mathcal{O}_{\Sigma^G^C} \cong \mathcal{O}_2, \quad \mathcal{I}_C \otimes \mathcal{K} \cong \mathcal{O}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \otimes \mathcal{K}.$$ 

The second example is the canonical $\lambda$-graph system for the Dyck shift $D_2$, that is not a sofic subshift. The subshift comes from automata theory and language theory (compare [1, 11]). Its alphabet $\Sigma$ consists of two kinds of four brackets: $\{., \},$ and $[..]$. The forbidden words consist of words that do not obey the standard bracket rules. Let $\mathcal{L}^{D_2}$ be the canonical $\lambda$-graph system for $D_2$. In [29], the K-groups of the symbolic matrix system for $\mathcal{L}^{D_2}$ have been computed. They are the K-groups for the associated $C^*$-algebra $\mathcal{O}_{\Sigma^{D_2}}$, so that we see $K_0(\mathcal{O}_{\Sigma^{D_2}}) \cong \mathbb{Z}^{\infty}$, and $K_1(\mathcal{O}_{\Sigma^{D_2}}) \cong 0$, where $\mathbb{Z}^{\infty}$ is the countable infinite sum of the group $\mathbb{Z}$. The $C^*$-algebra $\mathcal{O}_{\Sigma^{D_2}}$ has a proper ideal.
The $\lambda$-graph system $\mathcal{L}^D_\lambda$ satisfies condition (II). Let $\mathcal{L}^{Ch(D_2)}_\lambda$ be the $\lambda$-graph subsystem of $\mathcal{L}^D_\lambda$, called the Cantor horizon $\lambda$-graph system of $D_2$ (see [16] for details). Then $\mathcal{L}^{Ch(D_2)}_\lambda$ is aperiodic and a minimal irreducible component of $\mathcal{L}^D_\lambda$. Hence the associated algebra $O_{\mathcal{L}^{Ch(D_2)}_\lambda}$ is a simple purely infinite $C^*$-algebra realized as a quotient of $O_{\mathcal{L}^D_\lambda}$ by an ideal corresponding to a saturated hereditary subset of $\mathcal{L}^D_\lambda$. In [16], its $K$-groups have been computed to be $K_0(O_{\mathcal{L}^{Ch(D_2)}_\lambda}) \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathcal{C}, \mathbb{Z})$, and $K_1(O_{\mathcal{L}^{Ch(D_2)}_\lambda}) \cong 0$, where $C(\mathcal{C}, \mathbb{Z})$ denotes the abelian group of all $\mathbb{Z}$-valued continuous functions on a Cantor discontinuum $\mathcal{C}$. As $\mathcal{L}^{Ch(D_2)}_\lambda$ is predecessor-separated, the algebra $O_{\mathcal{L}^{Ch(D_2)}_\lambda}$ is generated by only the four partial isometries $S_\alpha, R, T$ corresponding to the brackets $(, )$, $[,]$ corresponding to the brackets $(,)$. Hence $O_{\mathcal{L}^{Ch(D_2)}_\lambda}$ is finitely generated, but its $K_0$-group is not finitely generated. This means that the algebra $O_{\mathcal{L}^{Ch(D_2)}_\lambda}$ is simple and purely infinite, but not semi-projective (compare [3]). Full details and its generalizations are seen in [16] and [20].

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References


