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HOMOMORPHISMS OF THE ALGEBRA OF LOCALLY INTEGRABLE FUNCTIONS ON THE HALF LINE

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Abstract

Let ϕ be a continuous nonzero homomorphism of the convolution algebra $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and also the unique extension of this homomorphism to $M_{\mathrm{loc}}(\mathbb{R}^+)$. We show that the map ϕ is continuous in the weak* and strong operator topologies on M_{loc}, considered as the dual space of $C_c(\mathbb{R}^+)$ and as the multiplier algebra of L^1_{loc}. Analogous results are proved for homomorphisms from $L^1[0,a)$ to $L^1[0,b)$. For each convolution algebra $L^1(\omega_1)$, ϕ restricts to a continuous homomorphism from some $L^1(\omega_1)$ to some $L^1(\omega_2)$, and, for each sufficiently large $L^1(\omega_2)$, ϕ restricts to a continuous homomorphism from some $L^1(\omega_1)$ to $L^1(\omega_2)$. We also determine which continuous homomorphisms between weighted convolution algebras extend to homomorphisms of L^1_{loc}. We also prove results on convergent nets, continuous semigroups, and bounded sets in M_{loc} that we need in our study of homomorphisms.

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1. Introduction

In this paper, we study the continuous homomorphisms of the convolution algebra $L^1_{\mathrm{loc}} = L^1_{\mathrm{loc}}(\mathbb{R}^+)$, the algebra of (almost everywhere equivalence classes of) locally integrable functions on the half line $\mathbb{R}^+ = [0,\infty)$. We also study the related convolution algebras $L^1[0,a)$, where $0 < a < \infty$, and often use properties of these algebras to study L^1_{loc}. We are particularly interested in extending homomorphisms between weighted convolution algebras to homomorphisms of L^1_{loc}, and restricting homomorphisms of L^1_{loc} to continuous homomorphisms between weighted convolution algebras. Since the structure of L^1_{loc} is much simpler than that of many of its weighted

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convolution subalgebras, the hope is to learn more about homomorphisms between weighted convolution algebras by considering them as restrictions of homomorphisms of L^1_{loc}. Some results along these lines are in the final section of this paper.

In [10], Ghahramani and McClure study automorphisms and derivations of L^1_{loc} and solve the extension and restriction problems. In particular, Ghahramani and McClure [10] show that every isomorphism between weighted convolution algebras and every derivation of a weighted convolution algebra can be extended to an isomorphism or derivation of L^1_{loc}. Surprisingly, not every homomorphism between weighted convolution algebras is a restriction of a homomorphism of L^1_{loc}, but we are able to determine precisely the class of homomorphisms that can be extended to L^1_{loc} (see Theorem 7.1).

L^1_{loc} becomes a Fréchet space under the collection of seminorms

$$\|f\|_a = \int_0^a |f(t)|\,dt, \tag{1.1}$$

for $0 < a < \infty$. It is convenient to think of the functions in $L^1[0, a)$ being defined on all of $\mathbb{R}^+ = [0, \infty)$, but equal to 0 off $[0, a)$. With this convention, $L^1[0, a)$ becomes a Banach space under the norm $\|f\|_a$ of formula (1.1). L^1_{loc} becomes a Fréchet algebra and $L^1[0, a)$ becomes a Banach algebra under the usual convolution multiplication. For x in $I = [0, \infty)$ or $[0, a)$, this multiplication is given by

$$f * g(x) = \int_0^x f(x-t)g(t)\,dt, \quad \text{for } x \in I. \tag{1.2}$$

When we consider functions on $[0, a)$ extended to be 0 on $[a, \infty)$, then $f * g(x)$ is given by formula (1.2) only for x in $[0, a)$, but is equal to 0 off $[0, a)$. Thus $L^1[0, a)$ is a continuously embedded subspace of L^1_{loc}, but it is not a subalgebra.

We let R_a be the restriction map from $[0, \infty)$ to $[0, a)$. That is, $R_a f(x) = f(x)$ for x in $[0, a)$ and is 0 for $x \geq a$. Thus R_a is a continuous algebra homomorphism from L^1_{loc} onto $L^1[0, a)$ with kernel

$$L^1_a = \{f \in L^1_{\mathrm{loc}} : \text{support } f \subseteq [a, \infty)\}.$$

Then $L^1[0, a)$ is isomorphic and homeomorphic to L^1_{loc}/L^1_a. Often the easiest way to prove results, particularly basic results, for L^1_{loc} is to first prove the analogous results for the spaces $L^1[0, a)$ and then 'lift' the results to L^1_{loc}.

The positive Borel function $\omega(x)$ on $[0, \infty)$ is a *weight* if both $\omega(x)$ and $1/\omega(x)$ are bounded on all intervals $[0, a)$. Then $L^1(\omega)$ is the subspace of L^1_{loc} composed of all f with the norm

$$\|f\| = \|f\|_\omega = \int_0^\infty |f(t)|\omega(t)\,dt$$

finite. Under this norm each $L^1(\omega)$ is a Banach space continuously embedded in the Fréchet space L^1_{loc}. We say that the weight $\omega(x)$ is an *algebra weight* when $\omega(x)$ is *submultiplicative* (that is, $\omega(x+y) \leq \omega(x)\omega(y)$), right continuous, and has $\omega(0) = 1$. In this case $L^1(\omega)$ is a subalgebra of L^1_{loc}. In fact, whenever $L^1(\omega)$ is a subalgebra of L^1_{loc}, one can replace $\omega(x)$ with an algebra weight $\bar{\omega}(x)$ for which $L^1(\bar{\omega})$ is just $L^1(\omega)$ under an equivalent norm [13, Theorem 2.1]. One of our underlying goals is to use L^1_{loc} to study $L^1(\omega)$, and to use $L^1[0,a)$ to study L^1_{loc}.

Good formulas characterizing important operators like derivations, homomorphisms, isomorphisms, and multipliers of the convolution algebras L^1_{loc}, $L^1[0,a)$, and $L^1(\omega)$ all involve measures (see, for instance, [19, 5, 13, 10]). Thus we need to consider the corresponding measure algebras $M_{\mathrm{loc}} = M_{\mathrm{loc}}(\mathbb{R}^+)$, $M[0,a)$, and $M(\omega)$. The Fréchet space M_{loc} consists of all locally finite complex Radon 'measures'; that is, linear combinations of sigma-finite positive Radon measures, on $\mathbb{R}^+$ under the collection of seminorms $\|\mu\|_a = |\mu|([0,a))$. Then $M[0,a)$ is the Banach space of finite complex Borel measures on $[0,a)$ or, equivalently, the Borel measures μ on the compact space $[0,a]$ for which $\mu\{a\} = 0$. As with functions, we often consider the measures in $M[0,a)$ as being defined on all of $\mathbb{R}^+$ with $|\mu|[a,\infty) = 0$. Similarly $M(\omega)$ is the Banach space of locally finite measures μ for which the norm $\|\mu\| = \|\mu\|_\omega = \int_{\mathbb{R}^+} \omega(t)\, d|\mu|(t)$ is finite.

We define the convolution for measures in M_{loc} and $M[0,a)$ in the usual way (see the next section), so that M_{loc} is a Fréchet algebra and $M[0,a)$ is a Banach algebra. If ω is an algebra weight, then $M(\omega)$ is a Banach algebra continuously imbedded in M_{loc}. We usually identify the function $f(t)$ with the measure $f(t)\,dt$, so that L^1_{loc}, $L^1[0,a)$, and $L^1(\omega)$ are all closed ideals in the corresponding measure algebras. We define the restriction maps in the obvious way, by $|R_a\mu|[a,\infty) = 0$, and we let M_a be the kernel of R_a. As with functions, R_a induces a (topological algebra) isomorphism from M_{loc}/M_a onto $M[0,a)$.

It is extremely useful to characterize the above measure algebras as dual spaces of appropriate spaces of continuous functions so that we have weak*-topologies on these spaces. If we identify $C_0[0,a)$ with the Banach space of continuous functions f on $[0,a]$ with $f(a) = 0$, then it follows from the Riesz representation theorem that $M[0,a)$ is (isometrically isomorphic to) the dual space of $C_0[0,a)$. If $\omega(x)$ is an algebra weight, then $M(\omega)$ is the dual space of $C_0(1/\omega)$, the Banach space of continuous functions h on $\mathbb{R}^+$ for which $h(x)/\omega(x)$ vanishes at ∞ and the norm $\|h\| = \sup\{|h(x)|/\omega(x) : x \geq 0\}$ is finite [13, Theorem 2.2]. We also consider M_{loc} as the dual space of the space $C_0(\mathbb{R}^+)$ of continuous functions with compact support in $\mathbb{R}^+ = [0,\infty)$.

The main results in this paper can be grouped into three parts. Sections 2–4 give results on the structure of the algebras L^1_{loc} and $L^1[0,a)$, together with M_{loc} and $M[0,a)$, that are needed for our study of homomorphisms and their extensions and restrictions. Sections 5 and 6 study the homomorphisms of these algebras, and Sections 7 studies extensions of homomorphisms from weighted convolution algebras to L^1_{loc} and the far more difficult problem of restricting homomorphisms of L^1_{loc} to weighted convolution algebras. Section 8 discusses applications to weighted convolution algebras and lists open problems suggested by our results.

2. Convergence and other basic properties

In this section, we recall the basic definitions for the algebras $L^1[0,a)$ and L^1_{loc} and their corresponding measure algebras. We prove analogues of the convergence results true for $L^1(\omega)$ and $M(\omega)$, when ω is an algebra weight. We consider M_{loc} as the dual space of $C_c(\mathbb{R}^+)$ under the duality

$$\langle \mu, h\rangle = \int_{\mathbb{R}^+} h(t)\, d\mu(t).$$

The same formula works for $M[0,a)$ and $C_0[0,a)$ if we follow the convention of extending functions and measures to be identically equal to 0 on $[a,\infty)$. For us, the most useful characterization of convolution in M_{loc} is given in [2, Equation (4.7.3)] and [10, page 55], where

$$\langle \mu * \nu, h\rangle = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} h(x+y)\, d\mu(x)\, d\nu(y) \quad \text{for all } h \text{ in } C_c(\mathbb{R}^+). \tag{2.1}$$

This reduces to the familiar formula $f*\mu(x) = \int_{[0,x)} f(x-t)d\mu(t)$ when we identify f in L^1_{loc} with the measure $f(t)\,dt$. On $M[0,a)$ we can either replace $\mathbb{R}^+$ with $[0,a)$ in (2.1) or consider the measures extended to vanish off $[0,a)$ and define convolution as the restriction of the measure defined by formula (2.1) to the interval $[0,a)$. An easy application of Fubini's theorem, as in [13, page 595], shows that convolution by μ on M_{loc} is the adjoint of the operator $\mu\hat{*}h(x) = \int_{\mathbb{R}^+} h(x+t)\, d\mu(t)$ on $C_c(\mathbb{R}^+)$. The analogous result holds for $M[0,a)$. So we have (compare [13, Lemma 3.1]):

LEMMA 2.1. *Convolution by a fixed measure in M_{loc} or $M[0,a)$ is a weak*-continuous linear map.*

It follows from the duality between $M[0,a)$ and $C_0[0,a)$, just as for locally compact groups, that $M[0,a)$ is the multiplier algebra of $L^1[0,a)$ (see [19, Remark 10]). From this it follows easily [10, Theorem 2.14] that M_{loc} is the multiplier algebra of L^1_{loc}. If ω is an algebra weight, then $M(\omega)$ is the dual space of $C_0(1/\omega)$ and the multiplier algebra of $L^1(\omega)$ (see [13, Theorem 2.2]). This is why we need the normalizations in our definitions of algebra weight. For a measure μ in M_{loc}, we use the standard notation $\alpha(\mu) = \inf(\operatorname{support}\mu)$, with $\inf(0) = \infty$. The key result about $\alpha(\mu)$ is the Titschmarsh Convolution Theorem [2, Theorem 4.7.22] $\alpha(\mu * \nu) = \alpha(\mu) + \alpha(\nu)$, for μ and ν in M_{loc} (or $M[0, a)$). This implies, in particular, that M_{loc} is an integral domain, and that μ in $M[0, a)$ is a divisor of zero if and only if $\alpha(\mu) > 0$.

When J is a linear subspace of $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ or $M_{\mathrm{loc}}(\mathbb{R}^+)$ and $0 \le b \le \infty$, we let

$$J_b = \{\mu \in J : \alpha(\mu) \ge b\} = \{\mu \in J : \operatorname{support}\mu \subseteq [b, \infty)\}.$$

Notice that $J_o = J$ and $J_\infty = \{0\}$. We abbreviate $L^1_{\mathrm{loc}}(\mathbb{R}^+)_b$ and $M_{\mathrm{loc}}(\mathbb{R}^+)_b$ by L^1_b and M_b, respectively. We use the same notation for subspaces of $L^1[0, a)$ and $M[0, a)$, but in this case $J_b = \{0\}$ for $b \ge a$. The spaces L^1_b and $L^1[0, a)$, and the spaces $L^1(\omega)_b$ when ω is an algebra weight, are closed ideals in the algebras L^1_{loc}, $L^1[0, a)$, and $L^1(\omega)$, respectively. They are called the *standard ideals*. It is a classical result [2, Theorem 4.7.58 (i)] that all closed ideals in $L^1[0, a)$ are standard. From this, one can conclude [10, Proposition 2.5 (a)] that all closed ideals in L^1_{loc} are also standard. The situation in $L^1(\omega)$ is not yet understood for all algebra weights $\omega(x)$.

We let δ_t in M_{loc} be the point mass at $t \ge 0$. Thus δ_0 is the identity, and convolution by δ_t is right translation by t. In $M[0, a)$ we use δ_t to stand for the restriction $R_a\delta_t$ of the measure δ_t to $[0, a)$. Thus $\delta_t = 0$ for $t \ge a$, and for all t, convolution by δ_t translates by t, and then truncates to $[0, a)$.

We are now ready to discuss weak* and metric convergence of bounded sequences and nets in $M[0, a)$ and M_{loc}. Recall that a subset of M_{loc} is *bounded* if it is bounded in all the seminorms $\|\mu\|_a = |\mu|[0, a)$, given by (1.1). The result for $M[0, a)$ is the following.

THEOREM 2.2. *Suppose that $\{\lambda_n\}$ is a bounded net in $M[0, a)$ and that λ belongs to $M[0, a)$. Suppose also that there is a ν in $M[0, a)$ with $\alpha(\nu) = 0$ for which $\lambda_\eta * \nu$ converges weak* to $\lambda * \nu$ (which would hold in particular if $\lambda_\eta = \lambda_n * \delta_0$ converged weak* to λ). Then we have*:

(a) *$\lambda_\eta * \mu$ converges weak* to $\lambda * \mu$ for all μ in $M[0, a)$;*
(b) *$\lambda_n * f$ converges in norm to $\lambda * f$ for all f in $L^1[0, a)$.*

REMARK. We do not normally have norm convergence if μ is not a function. For instance, $\delta_t = \delta_t * \delta_0$ converges weak* to δ_0 as $t \to 0^+$, but $\|\delta_t - \delta_0\| = 2$ for all $0 < t < a$.

PROOF. The proof of (a) is similar to the proof of the analogous result for $L^1(\omega)$ (see [13, Lemma 3.2]). First, it is enough to show that $\{\lambda_n\}$ converges weak* to λ. Then the weak*-continuity of multiplication by μ, given by Lemma 2.1, shows that $\lambda_n * \mu$ converges weak* to $\lambda * \mu$. Since every bounded net in $M[0, a)$ has a weak*-convergent subnet, we only need to show that if $\{\lambda'_n\}$ is a subnet of $\{\lambda_n\}$ with weak*-limit λ', then

$\lambda' = \lambda$. It follows from the weak*-continuity of multiplication, that $\lambda'_n * \nu$ converges to $\lambda' * \nu$, so that $\lambda * \nu = \lambda' * \nu$. Since $\alpha(\nu) \neq 0$, ν is not a divisor of zero, so we have $\lambda = \lambda'$ as required.

Now fix a function f in $L^1[0, a)$. Since we already know that $\lambda_n * f$ converges weak* to $\lambda * f$, it is enough to show that convolution by f is a compact operator from $M[0, a)$ to $L^1[0, a)$. It is a standard result that convolution by f is a compact operator on $L^1[0, a)$ (see for instance, [17, Theorem 2.5, pages 40 and 66]). From this, one shows that convolution by f is also a compact operator $M[0, a)$, exactly as in [7, Lemma 3.1].

The analogous result for M_{loc} follows easily from the above theorem. We do not even need to assume $\alpha(\nu) = 0$. □

THEOREM 2.3. *Suppose that* $\{\lambda_n\}$ *is a bounded net in* M_{loc}. *If there is a measure* $\nu \neq 0$ *for which* $\lambda_n * \nu$ *converges to* $\lambda * \nu$ *weak* in* M_{loc}, *then we have*

(a) $\lambda_n * \mu$ *converges weak* to* $\lambda * \mu$ *for all* μ *in* M_{loc};

(b) $\lambda_n * f$ *converges to* $\lambda * f$ *in the Fréchet topology of* L^1_{loc} *for all* f *in* L^1_{loc}.

PROOF. Suppose $\alpha(\nu) = b$ and let $\nu = \delta_t * \tau$, so that $\alpha(\tau) = 0$. Since convolution with δ_t is a linear homeomorphism from M_{loc} onto M_t in the weak* topology (and in the Fréchet topology), we have $\lambda_n * \tau$ converges weak* to $\lambda * \tau$. Thus there is no loss of generality in assuming $\alpha(\nu) = 0$.

It follows directly from the definitions that a net in M_{loc} is bounded in M_{loc}, or converges in the weak* topology or the Fréchet topology on M_{loc} if and only if its restrictions to all $[0, b)$ are bounded, converge weak*, or converge in norm in $M[0, b)$. Thus, the result follows from Theorem 2.2. □

In $M[0, a)$, unlike M_{loc}, the requirement that $\alpha(\nu) = 0$ is necessary. For instance, if $\{\lambda_n\}$ is any net in $M[0, a)$ with $\alpha(\lambda_n) + \alpha(\nu) \geq a$, then all $\lambda_n * \nu = 0$.

3. Semigroups

It has long been clear (see for instance [6, 13]) that the properties of homomorphisms between convolution algebras on $\mathbb{R}^+$ depend heavily on the properties of related convolution semigroups. In this section we prove the required results about continuity and support of convolution semigroups in M_{loc} and $M[0, a)$.

Suppose that $\{\mu_t\}_{t \geq 0}$ is a semigroup under convolution in M_{loc}. We identify μ_t with the semigroup of operators $f \mapsto \mu_t * f$ on L^1_{loc}. Thus we say that $\{\mu_t\}$ is (*strongly*) *continuous* (on L^1_{loc}) if $\mu_t * f$ is continuous in the metric topology on L^1_{loc} for t in $\mathbb{R}^+ = [0, \infty)$ and all f in L^1_{loc}. The standard Banach space result [18, Theorem 10.5.5] that a semigroup of operators is (strongly) continuous, provided that it is strongly continuous at $t = 0$, extends readily to semigroups on L^1_{loc}.

Similarly, we consider semigroups $\{\mu_t\}_{t\geq 0}$ in $M[0, a)$ as acting by convolution on $L^1[0, a)$. The main difference is that in $M[0, a)$ we can have some $\mu_{t_0} = 0$. In the presence of any kind of continuity this implies that there is a $b > 0$ for which $\mu_t = 0$ if and only if $t \geq b$. In this case we say that μ_t is a *nilpotent* semigroup of order b. For instance, the semigroup $\{\delta_t\}$ is nilpotent of order a in $M[0, a)$.

We now give the basic continuity results, first for M_{loc} and then for $M[0, a)$.

THEOREM 3.1. *Suppose that $\{\mu_t\}$ is a convolution semigroup in M_{loc} and is bounded for t in some interval about 0. If there is a $\nu \neq 0$ in M_{loc} for which $\mu_t * \nu$ is right continuous in the weak*-topology at some t in $[0, \infty)$, then we have*:

(a) *μ_t is a continuous semigroup on L^1_{loc};*
(b) *μ_t is a weak*-continuous semigroup on M_{loc}.*

PROOF. Suppose that $\mu_t * \nu$ is right continuous in the weak*- topology at $t = b \geq 0$. Then

$$\text{weak*-}\lim_{t\to 0^+} \mu_t * (\mu_b * \nu) = \text{weak*-}\lim_{t\to 0^+} (\mu_{t+b} * \nu) = \mu_b * \nu.$$

It then follows from Theorem 2.3 (b) that $\mu_t * f$ is right continuous at $t = 0$ in the metric topology on L^1_{loc}. This means that (convolution by) μ_t is a strongly continuous semigroup on L^1_{loc}. By Theorem 2.3 (a) we have that, for all λ in M_{loc}, $\mu_t * \lambda$ is a weak*-continuous function of t on $[0, \infty)$. This completes the proof. $\square$

The statement that μ_t is weak*-continuous on M_{loc} means that $\mu_t * \lambda$ is weak*-continuous for all λ in M_{loc}. By Theorem 2.3, this is equivalent to μ_t itself being weak*-continuous for t in $[0, \infty)$. On the other hand, μ_t is rarely a strongly continuous semigroup on all of M_{loc}, since this implies that $\mu_t = \mu_t * \delta_0$ is continuous in the Fréchet space topology on M_{loc}. From now on we will say $\{\mu_t\}$ is a *continuous semigroup* if it satisfies the conditions in Theorem 3.1.

We now give the analogous continuity result for semigroups in $M[0, a)$. The proof is the same except that, since we use Theorem 2.2 instead of Theorem 2.3, we need to add the restrictions that $\alpha(\nu) = 0$ and that $\mu_t * \nu$ is weak*-continuous at 0.

THEOREM 3.2. *Suppose that μ_t is a convolution semigroup in $M[0, a)$ and is bounded near 0. If there is a ν in $M[0, a)$ with $\alpha(\nu) = 0$ for which $\mu_t * \nu$ is weak*-continuous at 0, then μ_t acts as a strongly continuous semigroup on $L^1[0, a)$ and as a weak*-continuous semigroup on $M[0, a)$.*

The results for the support of semigroups in M_{loc} are the same as for $M(\omega)$. One can use the proofs of [13, Theorem 4.3] or [14, Lemma 3.1], both of which are adapted from proofs in [6]. The basic result for M_{loc} is the following.

THEOREM 3.3. *Suppose that μ_t is a strongly continuous convolution semigroup on L^1_{loc}. Then there is a real number $A \geq 0$ for which $\alpha(\mu_t) = At$ for all $t \geq 0$.*

When $\alpha(\mu_t) = At$, we say that A is the *character* of the semigroup $\{\mu_t\}$. For semigroups $\{\mu_t\}$ in $M[0, a)$, the statement $\alpha(\mu_t) \geq a$ means $\mu_t = 0$. So if $\{\mu_t\}$ has strictly positive character A, then μ_t is a nilpotent semigroup of order a/A. Below is a precise statement of the analogous result in $M[0, a)$.

THEOREM 3.4. *Suppose that $\{\mu_t\}$ is a continuous semigroup in $M[0, a)$. Then there is a real number $A \geq 0$ for which $\alpha(\mu_t) = At$ when $At < a$, and $\mu_t = 0$ for $At \geq a$.*

4. Boundedness

The most substantial results on homomorphisms of L^1_{loc} involve restrictions of homomorphisms from L^1_{loc} to weighted convolution algebras and extensions of homomorphisms between weighted convolution algebras to L^1_{loc}. We also need to consider the relation between nets and semigroups in M_{loc} and in algebras $M(\omega)$. The most important results for nets and semigroups involve bounded nets, and semigroups which are bounded near 0. Also, a continuous linear map is bounded. Because of this, we can base many of our results comparing L^1_{loc} and weighted convolution algebras on a comparison of bounded sets. The basic result for bounded sets is the following.

THEOREM 4.1. *A subset B of M_{loc} is bounded in M_{loc} if and only if there is an algebra weight ω for which B is a bounded subset of $M(\omega)$.*

We actually prove stronger results in each direction. When we start with subsets of $M(\omega)$, we do not require ω to be an algebra weight. When we construct $M(\omega)$, we can insure that $\omega(x)$ is not only an algebra weight, but also has additional useful properties. The next result gives the easier direction, which starts with $M(\omega)$.

THEOREM 4.2. *If B is a bounded subset of $M(\omega)$ for some weight $\omega(x)$, then B is a bounded subset of M_{loc}.*

PROOF. Our definition of weights requires $\omega(x)$ to be bounded away from 0 on each $[0, a)$. Fix a, and let $\omega(x) \geq k$ on $[0, a)$. Then we have

$$\|\mu\|_\omega = \int_{\mathbb{R}^+} \omega(t)\, d|\mu|(t) \geq \int_{[0,a)} \omega\, d|\mu| \geq k|\mu|([0, a)) = k\|\mu\|_a.$$

Thus any set bounded in the norm of $M(\omega)$ is also bounded in each of the seminorms that define the topology of M_{loc}. $\square$

Suppose that $a > 0$. We say that the weight $\omega(x)$ is *a-semiconvex* if $\omega(x)$ is continuous on $[0, \infty)$ and $\omega(x + a)/\omega(x)$ is (weakly) decreasing on $[0, \infty)$. When $a = 1$, we say that $\omega(x)$ is *semiconvex*. The weight $\omega(x)$ is said to be convex if it is a-semiconvex for every $a > 0$ [2, page 520]. The reason for the terminology is that if $\omega(x) = e^{-\eta(x)}$, then it can be shown that $\omega(x)$ is a-semiconvex for all $a > 0$ if and only if $\eta(x)$ is a convex function. The basic facts we need about a-semiconvex weights are collected in the following lemma.

LEMMA 4.3. *Suppose that $\omega(x)$ is an a-semiconvex weight and let*

$$K = K_\omega = \sup_{x,y \le a} \frac{\omega(x + y)}{\omega(x)\omega(y)}.$$

Then we have:

(a) $\omega(x + y) \le K\omega(x)\omega(y)$ *for all x and y in* $\mathbb{R}^+$;
(b) $\left(\lim_{x\to\infty} \omega(x)^{1/x}\right)^a = \lim_{x\to\infty} \omega(x + a)/\omega(x)$;
(c) *The weights $\omega(x + b)$ are all a-semiconvex for $b \ge 0$*;
(d) *If $\omega(x)$ is* (*weakly*) *decreasing, then* $\min(1, K\omega(x))$ *is an algebra weight equivalent to $\omega(x)$*.

PROOF. If $a \le x \le y$, it follows from our hypotheses that

$$\frac{\omega(y)}{\omega(x)} \le \frac{\omega(y - a)}{\omega(x - a)}. \tag{4.1}$$

Part (a) now follows easily from (4.1) (for the details see the proof of Theorem 2.2 in [11, pages 535–536]).

Let $\omega_1(x) = K\omega(x)$. Both limits in (b) are unchanged if $\omega(x)$ is replaced by $\omega_1(x)$. It follows from part (a) that $\omega_1(x)$ is submultiplicative. Therefore $\lim_{x\to\infty} \omega(x)^{1/x}$ exists, so that both limits in (b) exist. A simple argument shows the equality of the two limits (for instance, see [1, Lemma 1.2 (i), page 81]). Part (c) is straightforward.

Suppose now that $\omega(x)$ is decreasing. Since $K\omega(x)$ is submultiplicative, we have $K\omega(0) \ge 1$. Since $K(\omega(x))$ is continuous, decreasing, and submultiplicative, so is $\min(1, K\omega(x))$ (compare [1, page 81]) and hence is an algebra weight equivalent to $\omega(x)$. This completes the proof. □

For an a-semiconvex weight, Lemma 4.3 (a) shows that $M(\omega)$ and $L^1(\omega)$ are algebras, and the norm satisfies $\|\mu * \nu\|_\omega \le K\|\mu\|_\omega\|\nu\|_\omega$. Lemma 4.3 (c) then shows that each $L^1(\omega(x+b)) = \left\{ f \in L^1_{\text{loc}} : \delta_b * f \in L^1(\omega) \right\}$ is also an algebra, and similarly for $M(\omega(x + b))$.

We say that a weight $\omega(x)$ is a *radical* weight if $\lim_{x\to\infty} \omega(x)^{1/x} = 0$. For a-semiconvex weights, Lemma 4.3 (b) says that $\omega(x)$ is radical if and only if

$\lim_{x\to\infty} \omega(x+a)/\omega(x) = 0$. Weights with some $\lim_{x\to\infty} \omega(x+a)/\omega(x) = 0$ are called *regulated* weights (see, for example, [1] or [2, page 520]). These are the weights for which best results are known for homomorphisms, semigroups, and convergence in $M(\omega)$ and $L^1(\omega)$ [9, 14, 7]. Replacing $\omega(x)$ with an equivalent weight does not change whether or not $\lim_{x\to\infty} \omega(x+a)/\omega(x) = 0$.

We are now ready for the other half of Theorem 4.1.

THEOREM 4.4. *Suppose that B is a bounded subset of M_{loc}. Then there is a decreasing semiconvex algebra weight ω for which B is a bounded subset of $M(\omega)$. Moreover we can require that ω be a radical weight.*

The analogous result for a-semiconvex weights for any fixed a is true with essentially the same proof. The construction we give is adapted from our construction in [11, Theorem 6.5].

PROOF. We construct a decreasing semiconvex weight $\omega(x)$ for which $\omega(x) \equiv 1$ for $0 \le x \le 1$. For such a weight

$$K = \sup_{x,y\le 1} \frac{\omega(x+y)}{\omega(x)\omega(y)} = \sup_{x,y\le 1} \omega(x+y) = 1.$$

This weight $\omega(x)$ is an algebra weight by Lemma 4.3 (a).

We first construct a weight $\omega_0(x)$, which is not necessarily an algebra weight, for which B is a bounded subset of $M(\omega_0)$. For each nonnegative integer n, choose a positive number P_n, with $P_1 \ge 2$, and with $|\mu|[n, n+1) \le P_n$ for all μ in B. Such a number exists since $|\mu|[n, n+1) \le |\mu|[0, n+1) = \|\mu\|_{n+1}$. Let $\omega_0(x)$ be a continuous weakly decreasing function on $[0,\infty)$ with $\omega_0(x) \equiv 1$ on $[0, 1]$ and $\omega_0(n) \le 2^{-n} P_n$ for each positive integer n (for instance, $\omega_0(x)$ could be piecewise linear). Then, for each positive integer n and each measure μ in B, we have

$$\int_{[n,n+1)} \omega_0(t)\, d|\mu|(t) \le \omega_0(n)|\mu|[n, n+1) \le \frac{1}{2^n}\,.$$

Hence, for each μ in B we have

$$\|\mu\|_{\omega_0} = \int_{\mathbb{R}^+} \omega_0(t)\, d|\mu|(t) \le P_0 + 1.$$

To finish the proof we construct a continuous, decreasing, semiconvex weight $\omega(x)$ satisfying $\omega(x) \le \omega_0(x)$ for all x and $\omega(x) \equiv 1$ for x in $[0, 1]$. The connection between $\omega(x)$ and $\omega_0(x)$ is given by the function $\lambda(x) = \min\left\{\omega_0(t)/\omega_0(t-1) : 1 \le t \le x\right\}$ defined for $x \ge 1$. It is clear that $\lambda(x)$ is a positive, decreasing function with $\lambda(1) = 1$. Also, $\lambda(x)$ is continuous on $[1,\infty)$ since $\omega_0(x)/\omega_0(x-1)$ is continuous on the same interval $[1,\infty)$.

We now define $\omega(x) \equiv 1$ on $[0, 1]$ and $\omega(x) = \lambda(x)\omega(x-1)$ for $x \geq 1$. This is an inductive definition giving the value of $\omega(x)$ on an interval $[a+1, a+2)$ in terms of its value on $[a, a+1)$. Also $\omega(x)$ is well-defined at $x = 1$ since $\lambda(1)\omega(1-1) = 1$. Since both $\lambda(x)$ and $\omega(x-1)$ are positive, continuous, and decreasing on $[1, \infty)$, so is $\omega(x)$. Since $\omega(x) \equiv 1$ on $[0, 1]$, we have $\omega(x)$ is positive, decreasing, and continuous on all $[0, \infty)$.

We also have $\omega(x) \leq \omega_0(x)$, since whenever $\omega(x-1) \leq \omega_0(x-1)$, we have

$$\omega(x) = \lambda(x)\omega(x-1) \leq \frac{\omega_0(x)}{\omega_0(x-1)}\,\omega(x-1) \leq \frac{\omega_0(x)}{\omega_0(x-1)}\,\omega_0(x-1) = \omega_0(x).$$

Since $\omega(x) \leq \omega_0(x)$ for all x, every measure has a smaller norm in $M(\omega)$ than in $M(\omega_0)$. Thus B is bounded in $M(\omega)$ as well as in $M(\omega_0)$. Finally $\omega(x+1)/\omega(x) = \lambda(x+1)$ for x in $[0, \infty)$ and is therefore a decreasing function.

If the weight $\omega(x)$ is not a radical weight, then we replace $\omega(x)$ with $\omega(x)e^{-x^2}$. This completes the proof of Theorem 4.4, and hence of Theorem 4.1 as well. □

We now apply the boundedness results in this section to nets and semigroups. The applications to homomorphisms are given in Sections 7 and 8.

THEOREM 4.5. *Suppose that ω is an algebra weight and that $\{\lambda_n\}$ is a bounded net in $M(\omega)$. Then $\{\lambda_n\}$ is a bounded net in M_{loc}. Moreover $\{\lambda_n\}$ converges weak* to λ in $M(\omega)$ if and only if it converges to λ weak* in M_{loc}.*

PROOF. The boundedness of $\{\lambda_n\}$ in M_{loc} follows from Theorem 4.2. By definition, $\{\lambda_n\}$ converges to λ weak* in $M(\omega)$ when

$$\langle \lambda_n, h\rangle \to \langle \lambda, h\rangle \tag{4.2}$$

for all h in $C_0(1/\omega)$. For weak* convergence in M_{loc}, we need (4.2) to hold for all h in $C_c(\mathbb{R}^+)$. Since $C_c(\mathbb{R}^+)$ is a dense subspace of $C_0(1/\omega)$ and $\{\lambda_n\}$ is a bounded net in $M(\omega)$, the two kinds of weak convergence to λ are equivalent (compare the proof of [8, Equation (2.2)]). □

THEOREM 4.6. *Suppose that $\{\lambda_n\}$ is a bounded net in M_{loc}, which converges weak* in M_{loc} to λ. Then there is a semiconvex algebra weight ω for which $\{\lambda_n\}$ is a bounded net that converges weak* to λ in $M(\omega)$.*

PROOF. This follows immediately from Theorems 4.4 and 4.5. □

Theorems 3.1 (a) and 4.5 imply that if $\{\mu_t\}$ is a semigroup in $M(\omega)$ for some algebra weight and is bounded for t in some interval about 0, then $\{\mu_t\}$ is a weak*-continuous semigroup in $M(\omega)$ if and only if it is strongly continuous on L^1_{loc}. We can, in fact, choose ω so that $\{\mu_t\}$ is strongly continuous.

THEOREM 4.7. *Suppose that $\{\mu_t\}$ is a weak*-continuous semigroup in M_{loc}. Then there is a decreasing semiconvex algebra weight ω for which $\{\mu_t\}$ is a strongly continuous semigroup in $M(\omega)$. Moreover, we can require that ω be a radical weight.*

PROOF. Since $\{\mu_t\}$ is weak*-continuous in M_{loc}, we have that $\{\mu_t\}_{t\le 1}$ is a bounded subset of M_{loc}. It then follows from Theorem 4.4 that there is a radical decreasing semiconvex algebra weight ω for which $\{\mu_t\}_{t\le 1}$ is a bounded subset of $M(\omega)$.

Since $\mu_t = (\mu_t/n)^n$ for all $t \ge 0$ and all positive integers n, it follows that all μ_t belong to the algebra $M(\omega)$ and that $\{\mu_t\}$ is bounded for t in any bounded subset of $[0, \infty)$. The weak*-continuity of μ_t in $M(\omega)$ now follows from Theorem 4.5. As we pointed out after the proof of Lemma 4.3, ω is a regulated weight. Hence, every weak*-continuous semigroup in $M(\omega)$ is actually strongly continuous (see [14, Theorem 2.8] or the proof of [9, Theorem 3.4]). □

5. Homomorphisms of $L^1[0, a)$

In this section we study properties of homomorphisms from $L^1[0, a)$ to $L^1[0, b)$. In the next section we prove, partly by using results from the present section, analogous results for homomorphisms of L^1_{loc}. One of our techniques is to extend the homomorphisms to the corresponding measure algebras, so we also need to look at properties of homomorphisms between these measure algebras. If $c = b/a$, then $\phi(f(x)) = (1/c) f(x/c)$ is an isometric isomorphism from $L^1[0, a)$ onto $L^1[0, b)$. We could thus simply study the endomorphisms of $L^1[0, 1)$, but the more general formulation is useful when considering L^1_{loc}.

The following useful lemma will eventually, in Theorem 5.4, be replaced with a more general result.

LEMMA 5.1. *Suppose that ϕ is a continuous homomorphism from $L^1[0, a)$ (or $M[0, a)$) into $L^1[0, b)$ (or $M[0, b)$). If ϕ does not vanish on $L^1[0, a)$, then $\alpha(\mu) = 0$ if and only if $\alpha(\phi(\mu)) = 0$.*

PROOF. It follows from the Titschmarsh Convolution Theorem [2, Theorem 4.7.22] that for any measure λ, we have $\alpha(\lambda^n) = n\alpha(\lambda)$. Thus if $\alpha(\mu) > 0$ for some μ in $M[0, a)$, then μ is nilpotent in the algebra $M[0, a)$. Hence, $\phi(\mu)$ is nilpotent in $M[0, b)$, so that $\alpha(\phi(\mu)) > 0$ as required.

For the other case, we first consider the situation where ϕ is a nonzero continuous homomorphism from $L^1[0, a)$ to $L^1[0, b)$ or $M[0, b)$. We suppose that $\alpha(f) = 0$, but $\alpha(\phi(f)) \ne 0$ and we arrive at a contradiction. If $\alpha(\phi(f)) > 0$, then $\phi(f)$ is nilpotent, so there is a positive integer n for which $\phi(f^n) = (\phi(f))^n = 0$. Now $\ker(\phi)$ must be a standard ideal $L^1[0, a)_c$ for some $0 \le c$, by [10, Proposition 2.5 (a)]. However,

$\ker(\phi)$ contains f^n with $\alpha(f^n) = n$, $\alpha(f) = 0$. Hence $c = 0$, that is, ϕ is identically zero, contradicting our hypotheses.

Now suppose that ϕ maps $M[0, a)$ to $M[0, b)$ and let $\alpha(\mu) = 0$. Choose some f in $L^1[0, a)$ with $\alpha(f) = 0$. By replacing ϕ with its restriction to $L^1[0, a)$, we know that $\alpha(\phi(f)) = 0$. Now $\mu * f$ is a function in $L^1[0, a)$ with $\alpha(\mu * f) = \alpha(\mu) + \alpha(f) = 0$. Hence $0 = \alpha(\phi(\mu * f)) = \alpha(\phi(\mu)) + \alpha(\phi(f)) = \alpha(\phi(\mu))$. This completes the proof. □

We are now ready to extend our homomorphisms to the corresponding space of measures.

THEOREM 5.2. *Suppose that ϕ is a nonzero continuous homomorphism from $L^1[0, a)$ to $L^1[0, b)$ or $M[0, b)$. Then ϕ has a unique extension to a homomorphism $\bar{\phi} : M[0, a) \to M[0, b)$. Moreover we have*:

(a) *ϕ is continuous with $\|\bar{\phi}\| = \|\phi\|$;*
(b) *If ϕ is an isomorphism, so is $\bar{\phi}$, and we have $\overline{\phi^{-1}} = (\bar{\phi})^{-1}$.*

PROOF. The proof we gave in [13, Theorem 3.4] for the analogous result for weighted convolution algebras works in this case with one small modification. Since $M[0, b)$ is not an integral domain, it is not enough to have $\phi(h) \neq 0$ to 'cancel' $\phi(h)$ from formulae as we did in [13]. Instead, we choose h in $L^1[0, a)$ with $\alpha(h) = 0$. It then follows from Lemma 2.1 that $\alpha(\phi(h)) = 0$. The Titschmarsh Convolution Theorem then tells us that $\phi(h)$ is not a divisor of zero in $M[0, b)$. The proof given in [13] can then be carried out with this h instead. □

Because of the uniqueness of the extension of ϕ from $L^1[0, a)$ to $M[0, a)$, we will henceforth designate the extension by the same symbol ϕ. Thus if $\phi : L^1[0, a) \to L^1[0, b)$ is a continuous nonzero homomorphism and μ is a measure in $M[0, a)$, we write $\phi(\mu)$ instead of $\bar{\phi}(\mu)$.

The following convergence theorem is very useful. When we speak of the strong operator topology on $M[0, b)$, we identify the measure μ in $M[0, b)$ with the bounded operator $f \mapsto \mu * f$ on $L^1[0, b)$.

THEOREM 5.3. *Suppose that ϕ is a continuous nonzero homomorphism from $L^1[0, a)$ to $L^1[0, b)$ or $M[0, b)$. If $\{\lambda_n\}$ is a bounded net in $M[0, a)$ with weak*-limit λ, then $(\phi(\lambda_n))$ converges to $\phi(\lambda)$ in both the strong operator topology and the weak*-topology on $M[0, b)$.*

PROOF. We know from Theorem 2.2, that $\{\lambda_n\}$ converges to λ in the strong operator topology on $M[0, a)$. Choose some h in $M[0, a)$ with $\alpha(h) = 0$. It then follows that $\phi(\lambda_n) * \phi(h) = \phi(\lambda_n * h)$ converges to $\phi(\lambda) * \phi(h)$ in the norm of $M[0, b)$.

Since $\alpha(\phi(h)) = 0$, by Lemma 5.1, another application of Theorem 2.2 tells us that $\phi(\lambda_n)$ converges to $\phi(\lambda)$ in the strong operator topology and the weak*-topology on $M[0, b)$. This completes the proof. □

The analogue of the above result for weighted convolution algebras only works for special weights, since the analogue [7, Theorem 2.3] of Theorem 2.2 only holds for special weights.

Suppose that ϕ is a continuous nonzero homomorphism from $L^1[0, a)$ to $L^1[0, b)$ or $M[0, b)$. Then $\mu_t = \phi(\delta_t)$ is a semigroup in $M[0, b)$, and is a (strongly) continuous semigroup by Theorems 3.2 and 5.3. Thus, by Theorem 3.4, there is an $A \geq 0$, which we call the character of μ_t, for which $\alpha(\mu_t) = At$. As always in $M[0, b)$, $\alpha(\lambda) \geq b$ means $\lambda = 0$. When the semigroup $\phi(\delta_t) = \mu_t$ has character A, we also say that the homomorphism ϕ has *character* A. The following theorem is the basic result on the character of homomorphisms.

THEOREM 5.4. *Suppose that ϕ is a continuous nonzero homomorphism from $L^1[0, a)$ to $L^1[0, b)$ or $M[0, b)$. If ϕ has character A, then we have*:

(a) $A \geq b/a$;

(b) *If μ belongs to $M[0, a)$, then $\alpha(\phi(\mu)) = A\alpha(\mu)$*;

(c) *$\phi(\mu) = 0$ if and only if $\alpha(\mu) \geq b/A$, so that $\ker(\phi) = L^1[0, a)_{b/A}$ and ϕ is one-one if and only if $A = b/a$.*

PROOF. Since δ_a is 0 in $M[0, a)$, we have $\mu_a = \phi(\delta_a)$ is 0 in $M[0, b)$. Thus $Aa = \alpha(\mu_a) \geq b$, or $A \geq b/a$.

When $\mu = 0$ in $M[0, a)$, part (b) follows from our definitions. So suppose that μ is a nonzero measure in $M[0, a)$ and let $\alpha(\mu) = c < a$. Let ν be a measure in $M[0, a)$ with $\mu = \delta_c * \nu$. This means that, on $[0, a - c)$, ν is the left translate of μ by a distance c. Thus $\alpha(\nu) = 0$, and hence, by Lemma 5.1, $\alpha(\phi(\nu)) = 0$. Thus we have

$$\alpha(\phi(\mu)) = \alpha(\phi(\delta_c) * \phi(\nu)) = \alpha(\mu_c) + \alpha(\phi(\nu)) = Ac = A\alpha(\mu).$$

This proves part (b).

Part (c) is an immediate consequence of part (b), since $\phi(\mu) = 0$ if and only if $\alpha(\phi(\mu)) = A\phi(\mu) \geq b$. □

Part (c) of the theorem essentially says that ϕ is a one-one homomorphism from $L^1[0, b/A)$ that has been extended in a trivial way to $L^1[0, a)$, by mapping measures and functions on $[b/A, a)$ to 0 in $M[0, b)$.

For weighted convolution algebras, a class of homomorphisms, called standard homomorphisms in [9], have a number of very 'nice' properties. Only for special weights are all homomorphisms known to be standard, but the natural translations of the properties in [9] do hold for $L^1[0, a)$.

THEOREM 5.5. *Let ϕ be a nonzero continuous homomorphism from $L^1[0,a)$ to $L^1[0,b)$ (and also let ϕ designate the extension of this map to $M[0,a)$). Then the following hold*:

(a) *If $L^1[0,a) * f$ is dense in $L^1[0,a)$, then $L^1[0,b) * \phi(f)$ is dense in $L^1[0,b)$;*
(b) *$\phi(\delta_t) = \mu_t$ is a strongly continuous semigroup on $L^1[0,b)$;*
(c) *The extension $\phi : M[0,a) \to M[0,b)$ is continuous in the norm, weak*, and strong operator topologies and is continuous from the weak* topology on $M[0,a)$ to the strong operator topology on $M[0,b)$;*
(d) *The map $T : C_0[0,b) \to C_0[0,a)$ given by $Th(x) = \langle h, \phi(\delta_x)\rangle$ is a bounded linear transformation whose adjoint is $\phi : M[0,a) \to M[0,b)$;*
(e) *For all h in $L^1[0,b)$, there exists an f in $L^1[0,a)$ and a g in $L^1[0,b)$ for which $h = \phi(f) * g$.*

PROOF. We start with the results that we know already. Since $\alpha(f) = 0$ if and only if $L^1[0,a) * f$ is dense in $L^1[0,a)$ (see, for instance, [2, Theorem 4.7.58 (i)]), part (a) is essentially a restatement of part of Lemma 5.1. As we pointed out, $\{\mu_t\}$ is a strongly continuous semigroup by Theorems 3.2 and 5.3. The norm continuity of $\phi : M[0,a) \to M[0,b)$ is given in the original extension theorem, Theorem 5.2.

Suppose that S is a function from the dual space E of a Banach space to the Hausdorff space X. It follows from the Krein-Smulian Theorem [4, Theorem V.5.7] that S is weak*-continuous if it transforms bounded weak*-convergent nets in E to convergent nets in X (compare [16, Lemma 4.1]). The continuity of ϕ from the weak* topology on $M[0,a)$ to both the weak* and strong-operator topologies on $M[0,b)$ now follows from Theorem 5.3. We postpone the proof of strong operator topology continuity until we have proven part (e).

Since $\phi : M[0,a) \to M[0,b)$ is weak* continuous, there is some bounded linear operator $T : C_0[0,b) \to C_0[0,a)$ for which $\phi = T^*$. To prove (d), we just need to show that T is given by the formula in (d). For every x in $[0,a)$ we have that

$$Th(x) = \langle Th, \delta_x\rangle = \langle h, T^*\delta_x\rangle = \langle h, \phi(\delta_x)\rangle,$$

so (d) is proved.

We now prove (e). Let $\{e_n\}_{n=1}^{\infty}$ be a bounded approximate identity in $L^1[0,a)$; for instance, e_n could be n times the characteristic function of $[0,1/n)$. Let $f_n = \phi(e_n)$. It follows from Theorem 5.3 that $\{f_n\}$ is a bounded approximate identity in $L^1[0,b)$. We now make $L^1[0,b)$ into a Banach module over $L^1[0,a)$ by defining the multiplication $f \cdot g = \phi(f) * h$. Under this multiplication $\{e_n\}$ is a bounded approximate identity for the module $L^1[0,b)$. Part (e) now follows directly from the module form of the Cohen Factorization Theorem (see [2, Theorem 2.9.24] or [3, Theorem 16.1]).

We complete the proof of the theorem by showing that $\phi : M[0,a) \to M[0,b)$ is continuous in the strong operator topologies. Suppose that $\{\lambda_n\}$ is a net in $M[0,a)$

that converges to λ in the strong operator topology. Let h belong to $L^1[0, b)$ and write $h = \phi(f) * g$, as permitted by (e). Then

$$\phi(\lambda_n) * h = \phi(\lambda_n * f) * g \to \phi(\lambda * f) * g = \phi(\lambda) * \phi(f) * g = \phi(\lambda) * h. \qquad \square$$

6. Homomorphisms of L^1_{loc}

In this section we prove results for homomorphisms of L^1_{loc} that are analogous to the results given in the previous section for $L^1[0, a)$.

Recall that, for $c > 0$, the restriction map $R_c : L^1_{\mathrm{loc}} \to L^1[0, c)$ induces an algebraic and topological isomorphism from L^1_{loc}/L^1_c onto $L^1[0, c)$. We frequently identify $L^1[0, c)$ with the quotient algebra. In this case R_c becomes the quotient map and the restriction $R_c f$ is the coset of f in L^1_{loc}/L^1_c. The analogous facts hold for M_{loc}.

Let $T : L^1_{\mathrm{loc}} \to L^1_{\mathrm{loc}}$. The usual characterization of continuity of linear maps between locally convex spaces tells us (compare [10, page 54]) that T is continuous if and only if for all $b > 0$ there is an $a > 0$ and a number $K = K(a, b) > 0$ for which

$$\|Tf\|_b \leq K\|f\|_a. \tag{6.1}$$

Whenever (6.1) holds we have

$$T(L^1_a) \subseteq L^1_b. \tag{6.2}$$

If (6.2) holds for a linear transformation T, then T induces a linear map $T_{a,b} : L^1[0, a) \to L^1[0, b)$ given by

$$T_{a,b}(R_a f) = R_b T f. \tag{6.3}$$

Moreover, T is continuous (or is an algebra homomorphism) if and only if all $T_{a,b}$, given by (6.2) and (6.3), are continuous (or are homomorphisms). Analogous results hold for linear maps from L^1_{loc} (or M_{loc}) to L^1_{loc} (or M_{loc}).

The following result shows how the integral domain property of L^1_{loc} and M_{loc} can often give 'better' results than are true for $L^1[0, a)$ and $M[0, a)$.

THEOREM 6.1. *Suppose that ϕ is a continuous homomorphism from L^1_{loc} (or M_{loc}) to L^1_{loc} (or M_{loc}). If ϕ does not vanish on L^1_{loc}, then ϕ is one-one.*

PROOF. First suppose that the domain of ϕ is L^1_{loc}. If ϕ were not one-one, then its kernel would have to be [10, Proposition 2.5 (a)] one of the standard ideals L^1_a for $a > 0$. Then ϕ would induce a one-one homomorphism from L^1_{loc}/L^1_a into M_{loc}. However, this is impossible, since L^1_{loc}/L^1_a has nilpotent elements and M_{loc} is an integral domain.

Now suppose that $\phi : M_{\text{loc}} \to M_{\text{loc}}$ and let μ be a nonzero measure in M_{loc}. Choose a function $f \neq 0$ in L^1_{loc}. We have already shown that $\phi(\mu * f) = \phi(\mu) * \phi(f) \neq 0$. Hence $\phi(\mu) \neq 0$ as required. $\square$

We now use Theorem 6.1 to prove the following analogue of Lemma 5.1 for L^1_{loc} and M_{loc}.

LEMMA 6.2. *Suppose that ϕ is a continuous homomorphism from L^1_{loc} (or M_{loc}) to L^1_{loc} (or M_{loc}). Suppose also that ϕ does not vanish on L^1_{loc}. Then for all μ we have that $\alpha(\mu) = 0$ if and only if $\alpha(\phi(\mu)) = 0$.*

PROOF. For the case where $\alpha(\mu) = 0$ we first consider $\phi : L^1_{\text{loc}} \to M_{\text{loc}}$. The case where the domain of ϕ is M_{loc} then follows exactly as in the proof of Lemma 5.1. We suppose that there is an f in L^1_{loc} with $\alpha(f) = 0$, but with $\alpha(\phi(f)) > 0$. Let $b = \alpha(\phi(f))$. Then $\phi^{-1}(M_{2b})$ is a closed ideal of L^1_{loc} that does not contain f. Since all ideals of L^1_{loc} are standard [10, Proposition 2.5 (a)], there is an $a > 0$ for which $L^1_a = \phi^{-1}(M_{2b})$. It follows from the Titschmarsh Convolution Theorem that $\alpha(\phi(f^2)) = 2\alpha(\phi(f)) = 2b$ and hence $f^2 \in L^1_a$. This requires that $\alpha(f^2) = 2\alpha(f) \geq a > 0$, which contradicts our assumption that $\alpha(f) = 0$.

When $\alpha(\mu) > 0$, it follows from (6.2), which holds for any continuous linear map, that $\alpha(\phi(\mu)) > 0$. This completes the proof of the theorem. $\square$

We now prove that we can extend homomorphisms from L^1_{loc} to M_{loc}. This extension theorem plays the same crucial role that it played for $L^1[0, a)$ in Section 5 and for weighted convolution algebras in [13] and subsequent papers.

THEOREM 6.3. *Suppose that ϕ is a continuous nonzero homomorphism from L^1_{loc} to L^1_{loc} or M_{loc}. Then ϕ has a unique extension to a homomorphism $\bar{\phi} : M_{\text{loc}} \to M_{\text{loc}}$. Moreover this extension is continuous.*

We prove Theorem 6.3 in two stages, starting with the following lemma.

LEMMA 6.4. *Suppose that $\phi : L^1_{\text{loc}} \to M[0, b)$ is a continuous nonzero homomorphism. Then ϕ has a unique extension to a homomorphism $\bar{\phi} : M_{\text{loc}} \to M[0, b)$. Moreover, this unique extension is continuous.*

PROOF. Since L^1_{loc} is a domain, ϕ cannot be one-one; its kernel is a nonzero ideal L^1_a. Then ϕ induces a continuous monomorphism ϕ_{ab} from $L^1[0, a) \approx L^1_{\text{loc}}/L^1_a$ into $M[0, b)$. By Theorem 5.2, we can extend ϕ_{ab} to a continuous homomorphism $\bar{\phi}_{ab} : M[0, a) \to M[0, b)$. We now define $\bar{\phi} = \overline{\phi_{ab}} \cdot R_a$.

For uniqueness choose h in L^1_{loc} with $\alpha(h) = 0$. Then $\alpha(\phi(h)) = \alpha(\phi_{ab}(R_a h)) = 0$. Hence $\phi(h)$ is not a divisor of zero in $M[0, b)$. Thus $\bar{\phi}(\mu)$ is uniquely defined by the formula $\bar{\phi}(\mu) * \phi(h) = \phi(\mu * h)$. $\square$

PROOF OF THEOREM 6.3. Uniqueness follows as in Lemma 6.4, with the simplification that $\phi(h)$ is never a divisor of zero for $h \neq 0$.

For each $b > 0$, Lemma 6.4 shows that $R_b\phi : L^1_{\mathrm{loc}} \to M[0, b)$ has a unique extension to a continuous homomorphism $\bar{\phi}_b : M_{\mathrm{loc}} \to M[0, b)$. By the uniqueness, these restrictions are compatible in the sense that if $a < b$, then $\bar{\phi}_a(\mu)$ is the restriction of $\bar{\phi}_b(\mu)$ to $[0, a)$. Thus we can define $\bar{\phi} : M_{\mathrm{loc}} \to M_{\mathrm{loc}}$ by specifying that $R_b\bar{\phi}(\mu) = \bar{\phi}_b(\mu)$ for all $b > 0$. Since all $R_b\bar{\phi}$ are homomorphisms, so is $\bar{\phi}$. Similarly, the continuity of $\bar{\phi}$ follows from the continuity of all $R_b\bar{\phi} = \bar{\phi}_b$. This completes the proof of the theorem. □

Just as in the proof of [13, Theorem 3.4], we can show that if $\|Tf\|_b \le K\|f\|_a$ for all f in L^1_{loc}, then $\|T\mu\|_b \le K\|\mu\|_a$, with the same K, for all μ in M_{loc}.

Because of the uniqueness of the extension $\bar{\phi} : M_{\mathrm{loc}} \to M_{\mathrm{loc}}$ of ϕ, we will henceforth use ϕ to designate both the original map on L^1_{loc} and its extension to M_{loc}. The next theorem follows from Theorem 2.3 in the same way that Theorem 5.3 followed from Theorem 2.2.

THEOREM 6.5. *Suppose that ϕ is a continuous nonzero homomorphism from L^1_{loc} to L^1_{loc} or M_{loc}. If $\{\lambda_n\}$ is a bounded net in M_{loc} with weak* limit λ, then $\phi(\lambda_n)$ converges to $\phi(\lambda)$ in both the weak* and strong- operator topologies on M_{loc}.*

When ϕ is a continuous nonzero homomorphism from L^1_{loc} to L^1_{loc} or M_{loc}, we say that ϕ has character A when the semigroup $\phi(\delta_t)$ has character A. The following is the basic result about characters of homomorphisms from L^1_{loc}.

THEOREM 6.6. *Suppose that ϕ is a continuous nonzero homomorphism from L^1_{loc} to L^1_{loc} or M_{loc}, and let ϕ have character A. Then we have*:

(a) *The character A is strictly positive*;

(b) $\alpha(\phi(\mu)) = A\alpha(\mu)$ *for all μ in M_{loc}.*

PROOF. Choose $a > 0$. By (6.2) we know that there is a $b > 0$ with $\phi(M_a) \subseteq M_b$. In particular, $\alpha(\phi(\delta_a)) = Aa \ge b$. This proves (a).

Suppose μ belongs to M_{loc} and let $a = \alpha(\mu)$. We can write $\mu = \delta_a * \nu$, where ν is a measure with $\alpha(\nu) = 0$. Hence, by Lemma 6.2,

$$\alpha(\phi(\mu)) = \alpha(\phi(\delta_a)) + \alpha(\phi(\nu)) = Aa + 0 = A\alpha(\mu).$$

This completes the proof of the theorem. □

We are now ready to prove the 'standardness' properties analogous to those in Theorem 5.5.

THEOREM 6.7. *Suppose that $\phi : L^1_{\mathrm{loc}} \to L^1_{\mathrm{loc}}$ is a continuous nonzero homomorphism. Then the following properties hold*:

(a) *If $L^1_{\mathrm{loc}} * f$ is dense in L^1_{loc}, so is $L^1_{\mathrm{loc}} * \phi(f)$*;
(b) *$\phi(\delta_t) = \mu_t$ is a strongly continuous semigroup on L^1_{loc}*;
(c) *The extension map $\phi : M_{\mathrm{loc}} \to M_{\mathrm{loc}}$ is continuous in the weak* and strong operator topologies*;
(d) *For all h in L^1_{loc}, there are functions f and g in L^1_{loc} for which $h = \phi(f) * g$.*

PROOF. The proofs of (a) and (b) are the same as for Theorem 5.5 (a) and (b). The proof of (d) is the same as the proof of Theorem 5.5 (e), except that we use the Fréchet module version of the Cohen Factorization Theorem [3, Theorem 26.2]. The strong-operator-topology continuity of ϕ now follows from the factorization in (d), just as in Theorem 5.5 (c). To complete the proof, we just need to show that ϕ is weak* continuous.

Since $C_c(\mathbb{R}^+)$, the predual of M_{loc}, is essentially the union of all $C_0[0, b)$, it is clear that all the restriction maps $R_b : M_{\mathrm{loc}} \to M[0, b)$ are weak* continuous. It also follows that $\phi : M_{\mathrm{loc}} \to M_{\mathrm{loc}}$ is weak* continuous if and only if all $R_b\phi : M_{\mathrm{loc}} \to M[0, b)$ are continuous. Let ϕ have character $A > 0$, and fix $b > 0$. Then it follows from Theorem 6.6 (b) that if we let $a = b/A$, then we have $\phi(M_a) \subseteq M_b$. Let $\phi_{ab} : M[0, a) \to M[0, b)$ be the induced homomorphism given by $\phi_{ab}R_a = R_b\phi$, as in (6.3). Then ϕ_{ab} is weak* continuous, by Theorem 5.5 (c). Hence $\phi_{ab}R_a$ is also weak* continuous. Thus all $R_b\phi$ are weak* continuous, forcing ϕ to be weak* continuous. This completes the proof. □

With a bit more effort we can prove analogues of the remainder of Theorem 5.5. Thus ϕ is the adjoint of the map $T : C_c(\mathbb{R}^+) \to C_c(\mathbb{R}^+)$ given by $Th(x) = \langle h, \phi(\delta_x)\rangle$, and ϕ is continuous from the weak* topology on M_{loc} to the strong operator topology.

7. Extensions and restrictions of homomorphisms

In this section we show that all continuous homomorphisms of L^1_{loc} restrict to continuous homomorphisms between weighted convolution algebras. We determine precisely which homomorphisms between weighted convolution algebras extend to homomorphisms of L^1_{loc}. In the restriction theorems (Theorems 7.4 and 7.7) we show that we can choose the domain or choose the range of the restriction in an essentially arbitrary manner.

These restriction and extension theorems let us prove results about homomorphisms of L^1_{loc} from results about homomorphisms between weighted convolution algebras, and vice versa.

THEOREM 7.1. *Suppose that $L^1(\omega_1)$ and $L^1(\omega_2)$ are subalgebras of L^1_{loc} and that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism of L^1_{loc}.*

(a) *If ϕ has a positive character, then it has a unique extension to a continuous homomorphism of L^1_{loc}.*

(b) *If the character of ϕ is 0, then no linear extension of ϕ to L^1_{loc} can be continuous.*

PROOF. Since ω_1 and ω_2 are weights and $L^1(\omega_1)$ and $L^1(\omega_2)$ are subalgebras of L^1_{loc}, we can replace ω_1 and ω_2 by 'essentially equivalent' weights without changing the algebras or their norm topologies [13, Theorem 2.1]. Therefore, without loss of generality, we can assume for simplicity that ω_1 and ω_2 are algebra weights.

Since $L^1(\omega_1)$ and $L^1(\omega_2)$ are dense subalgebras of L^1_{loc}, the map ϕ can be continuously extended if and only if $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is continuous in the relative L^1_{loc} topologies on $L^1(\omega_1)$ and $L^1(\omega_2)$. Moreover, when the continuous extension exists, it is necessarily unique. The theorem will thus be a consequence of the following lemma.

LEMMA 7.2. *Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism of character $A \geq 0$.*

(a) *If $A > 0$, then ϕ is continuous in the (relative) L^1_{loc} topologies on $L^1(\omega_1)$ and $L^1(\omega_2)$.*

(b) *If $A = 0$, then ϕ is discontinuous in the L^1_{loc} relative topologies.*

PROOF. We first consider the case where ϕ has character 0. For all $a > 0$, there is an f in $L^1(\omega_1)$ with $\alpha(f) > a$ and $\alpha(\phi(f)) = 0$ [13, Theorem 4.11]. Thus $\phi(L^1(\omega_1)_a)$ cannot be a subset of any $L^1(\omega_2)_b$. Hence ϕ cannot satisfy the condition for L^1_{loc}-topology continuity given in (6.1) and (6.2).

Now we assume that the character A is strictly positive. Then for all f in $L^1(\omega_1)$ we have, by [15, Theorem 5.8] and [13, Theorem 4.9 (a)], that $\alpha(\phi(f)) = A\alpha(f)$. Hence, given $b > 0$, we can find an $a > 0$ for which

$$\phi(L^1(\omega_1)_a) \subseteq L^1(\omega_2)_b. \tag{7.1}$$

In fact, (7.1) holds if and only if $b \geq aA$. As with L^1_{loc}, the restriction map $R_a : L^1(\omega_1) \to L^1[0, a)$ induces a homeomorphic algebra isomorphism (though normally not an isometry) from $L^1(\omega_1)/L^1(\omega_1)_a$ onto $L^1[0, a)$. We therefore identify $L^1[0, a)$ with the quotient space and R_a with the quotient. Of course the analogous results hold for $R_b : L^1(\omega_2) \to L^1(\omega_2)_b$. Thus, just as for formula (6.3), ϕ induces a bounded linear transformation $\phi_{ab} : L^1[0, a) \to L^1[0, b)$ satisfying $\phi_{ab} R_a f = R_b \phi f$ for all f in $L^1(\omega_1)$. Hence for all f in $L^1(\omega_1)$ we have

$$\|\phi f\|_b = \|R_b \phi f\| = \|\phi_{ab} R_a f\| \leq \|\phi_{ab}\| \|R_b f\| = \|\phi_{ab}\| \|f\|_a.$$

Therefore ϕ satisfies the condition for L^1_{loc}-topology continuity given by (6.1). This completes the proof of the lemma and of Theorem 7.1. $\square$

In the above proof the fact that ϕ is a homomorphism of positive character is only used to prove (7.1) and hence works for any bounded linear map satisfying this formula. Essentially the same proof works for $\phi : L^1(\omega_1) \to M(\omega_2)$, except that here we need to assume that ω_2 is an algebra weight. This is because replacing ω_2 with an essentially equivalent weight preserves $L^1(\omega_2)$, but can change $M(\omega_2)$. The precise result is as follows.

THEOREM 7.3. *Suppose that $L^1(\omega_1)$ is a subalgebra of L^1_{loc}, that ω_2 is an algebra weight, and that $\phi : L^1(\omega_1) \to M(\omega_2)$ is a continuous nonzero homomorphism. Then ϕ is continuous in the relative L^1_{loc} topologies, and hence has an extension to a continuous homomorphism from L^1_{loc} to M_{loc}, if and only if ϕ has positive character. When the extension exists, it is unique.*

In connection with the above theorem, it is worth noting that the class of homomorphisms of positive character is sufficiently large to include all continuous nonzero endomorphisms of radical $L^1(\omega)$ [13, Theorem 4.7].

We now prove our restriction theorems. First notice that, for every weight ω, we have $L^1(\omega)$ continuously imbedded in L^1_{loc} and M_{loc}, and $M(\omega)$ continuously imbedded in M_{loc}. Thus, it follows from the closed graph theorem that restrictions of a linear map continuous in the Fréchet topologies of L^1_{loc} and M_{loc} to weighted convolution algebras is automatically continuous. We start by specifying the domain. In this case the map need not even be a homomorphism.

THEOREM 7.4. *Suppose that $\phi : L^1_{\mathrm{loc}} \to L^1_{\mathrm{loc}}$ is a continuous linear map. If ω_1 is any weight on R^+, then there is a decreasing continuous semiconvex algebra weight ω_2 for which $\phi(L^1(\omega_1)) \subseteq L^1(\omega_2)$, so that ϕ restricts to a continuous linear map from $L^1(\omega_1)$ to $L^1(\omega_2)$. The analogous result holds for $\phi : L^1_{\mathrm{loc}} \to M_{\mathrm{loc}}$ with $\phi(L^1(\omega_1)) \subseteq M(\omega_2)$.*

PROOF. For definiteness, we consider the case where ϕ is a continuous linear map on L^1_{loc}. Let B be the closed unit ball in $L^1(\omega_1)$. It follows from Theorem 4.2 that B is a bounded subset of L^1_{loc}. Since ϕ is continuous, this implies that $\phi(B)$ is a bounded subset of $L^1_{\mathrm{loc}} \subseteq M_{\mathrm{loc}}$. Theorem 4.4 then guarantees a weight ω_2 with the specified properties for which $\phi(B) \subseteq M(\omega_2) \cap L^1_{\mathrm{loc}} = L^1(\omega_2)$. Since $L^1(\omega_1)$ is the linear span of B, this implies that $\phi(L^1(\omega_1)) \subseteq L^1(\omega_2)$ as required. The case of $\phi : L^1_{\mathrm{loc}} \to M_{\mathrm{loc}}$ is similar, except that we do not have $\phi(B) \subseteq L^1_{\mathrm{loc}}$ so we can only conclude $\phi(L^1(\omega_1)) \subseteq M(\omega_2)$. This completes the proof. $\square$

We know from Theorem 6.3 that every continuous nonzero homomorphism $\phi : L^1_{\mathrm{loc}} \to M_{\mathrm{loc}}$ has a unique extension to a continuous homomorphism of M_{loc}. On the other hand, many of our results on homomorphisms of M_{loc} require that the homomorphisms not vanish on L^1_{loc} (see, for instance, Lemma 6.2). So the class of homomorphisms we consider are nonzero continuous homomorphisms from L^1_{loc} to M_{loc} and their continuous extensions to M_{loc}; hence we usually identify $\phi : L^1_{\mathrm{loc}} \to M_{\mathrm{loc}}$ with its extension. Because of [13, Theorem 3.4] we have the analogous unique extension result and use the same convention of identifying a nonzero continuous homomorphism $\phi : L^1(\omega_1) \to M(\omega_2)$ with its extension to $M(\omega_1)$.

We will need the following extension of a result in [13].

THEOREM 7.5. *Suppose that $\{\mu_t\}$ is a strongly continuous semigroup in $M(\omega_2)$, where ω_2 is an algebra weight. Suppose also that $\omega_1(t)$ is an algebra weight for which $\|\mu_t\|_{\omega_2}/\omega_1(t)$ is bounded. Then there exists a unique nonzero continuous homomorphism $\phi : L^1(\omega_1) \to M(\omega_2)$ for which $\phi(\delta_t) = \mu_t$.*

PROOF. The existence is given in [13, Theorem 3.17]. The map ϕ is given by the strong Bochner integral $\phi(f) = \int_{\mathbb{R}^+} f(t)\mu_t\,dt$, that is, $\phi(f) * g = \int_{\mathbb{R}^+} f(t)\mu_t * g\,dt$, is a Bochner integral in $M(\omega_2)$, for all g in $L^1(\omega_2)$.

To show that ϕ is nonzero, we first consider the case where $\lim_{t\to\infty}\omega_1(t)^{1/t} < 1$. In this case $u(t) \equiv 1$ belongs to $L^1(\omega_1)$. Let $-A$ be the generator of the semigroup μ_t. Then A is a closed densely defined operator on $L^1(\omega_2)$ and is invertible with $A^{-1} = \int_{\mathbb{R}^+} \mu_t\,dt = \int_{\mathbb{R}^+} u(t)\mu_t\,dt$ is a strong Bochner integral [13, page 601]. Hence A^{-1} is (the operator on $L^1(\omega_2)$ of convolution by) $\phi(u)$, so $\phi(u) \neq 0$. When $\lim_{t\to\infty}\omega_1(t)^{1/t} < e^r$, one replaces the formula for $-A^{-1}$ with the resolvent formulas $(\lambda + A)^{-1} = \int_{\mathbb{R}^+} e^{-\lambda t}\mu_t\,dt = \phi(e^{-\lambda t})$ for $\lambda > r$ [13, Lemma 11.5.1].

Now suppose that $\Psi : L^1(\omega_1) \to M(\omega_2)$ is another nonzero continuous homomorphism with $\Psi(\delta_t) = \mu_t$. Then we again have [13, Theorem 3.6] that $\Psi(f) = \int f(t)\mu_t\,dt$ is a strong Bochner integral. Then $\phi = \Psi$, and the proof is complete. $\square$

The condition that $\|\mu_t\|_{\omega_2}/\omega_1(t)$ is bounded is necessary as well as sufficient since we must have $\|\mu_t\|_{\omega_2} \leq \|\phi\|\|\delta_b\|_{\omega_1} = \|\phi\|\omega_1(t)$. The natural choice of $\omega_1(t)$ would be $\omega'(t) = \|\mu_t\|_{\omega_2}$, but we cannot be sure that $\omega'(t)$ is an algebra weight in our sense of the term. However, we can always find an algebra weight $\omega_1(t)$ for which both $\omega'(t)/\omega_1(t)$ and $\omega_1(t)/\omega'(t)$ are bounded (see [13, page 610] and [12, Lemma 2.1]). In this case $M(\omega_1)$ and $M(\omega')$ are the same algebras with equivalent norms. We can use Theorem 7.4, to get an application of Theorem 7.5 to L^1_{loc}. We will need this application in our other restriction theorem.

LEMMA 7.6. *Suppose that ϕ and Ψ are continuous nonzero homomorphisms from L^1_{loc} to M_{loc}. If $\phi(\delta_t) = \Psi(\delta_t)$ for all t, then $\phi = \Psi$.*

PROOF. Let $\omega(t)$ be a radical algebra weight. By Theorem 7.4, there are algebra weights ω_1 and ω_2 with $\phi(L^1(\omega)) \subseteq M(\omega_1)$ and $\Psi(L^1(\omega)) \subseteq M(\omega_2)$. Then ω_1 and

ω_2 are algebra weights, which must also be radical weights, so that both $M(\omega_1)$ and $M(\omega_2)$ are subalgebras of $M(\omega_1\omega_2)$, and $\omega_1\omega_2$ is an algebra weight. It follows from Theorems 4.5 and 6.7 (b) that $\phi(\delta_t) = \Psi(\delta_t)$ is a weak*-continuous semigroup in $M(\omega_1\omega_2)$. Now let $\omega_3(t) = e^{-t}\omega_1(t)\omega_2(t)$. Just as in the proof of Theorem 4.7, we have $\phi(\delta_t) = \Psi(\delta_t)$ is a strongly continuous semigroup in $M(\omega_3)$. Thus ϕ and Ψ are continuous maps from $L^1(\omega)$ to $M(\omega_3)$, for which $\phi(\delta_t) = \Psi(\delta_t)$ is a strongly continuous semigroup. It then follows from Theorem 7.5 that ϕ and Ψ agree on $L^1(\omega)$. Since $L^1(\omega)$ is dense in L^1_{loc}, this implies that $\phi = \Psi$, and the proof is complete. $\square$

We now have our other restriction theorem, in which we specify the range instead of the domain.

THEOREM 7.7. *Let $\phi : L^1_{\text{loc}} \to M_{\text{loc}}$ be a continuous nonzero homomorphism and let $\mu_t = \phi(\delta_t)$. If ω_2 is an algebra weight for which μ_t is a strongly continuous semigroup in $M(\omega_2)$ (such weights exist by Theorem 4.7), then for any algebra weight $\omega_1(t)$ with $\|\mu_t\|_{\omega_2}/\omega_1(t)$ bounded, we have that ϕ restricts to a continuous nonzero homomorphism from $L^1(\omega_1)$ to $M(\omega_2)$. Conversely, if $\phi(L^1(\omega_1)) \subseteq M(\omega_2)$, then $\|\mu_t\|_{\omega_2}/\omega_1(t)$ must be bounded.*

PROOF. By Theorem 7.5, there is a continuous nonzero homomorphism $\Psi : L^1(\omega_1) \to M(\omega_2)$ with $\Psi(\delta_t) = \mu_t$. It follows from Theorem 6.6 (a) that $\{\mu_t\}$, and hence Ψ, have positive character. Thus, by Theorem 7.3, Ψ can be extended by continuity to a homomorphism $\bar{\Psi} : L^1_{\text{loc}} \to M_{\text{loc}}$. Since $\phi(\delta_t) = \bar{\Psi}(\delta_t) = \mu_t$, it follows,from Lemma 7.6 that $\phi = \bar{\Psi}$. Thus ϕ restricts to Ψ on $L^1(\omega_1)$. Conversely if ϕ restricts to a continuous homomorphism from $L^1(\omega_1)$ to $M(\omega_2)$, then

$$\|\mu_t\|_{\omega_2} = \|\phi(\delta_t)\|_{\omega_2} \leq \|\phi\|\|\delta_t\|_{\omega_1} = \|\phi\|\omega_1(t).$$

This completes the proof. $\square$

A proof very similar to that above gives the following existence result, which complements the uniqueness in Lemma 7.6.

THEOREM 7.8. *Suppose that $\{\mu_t\}$ is a strongly continuous semigroup of positive character in M_{loc}. Then there exists a unique continuous nonzero homomorphism $\phi : L^1_{\text{loc}} \to M_{\text{loc}}$ with $\phi(\delta_t) = \mu_t$.*

PROOF. Uniqueness follows from Lemma 7.6. For existence, choose, by Theorem 4.7, an algebra weight ω_2 for which μ_t is a continuous semigroup in $M(\omega_2)$. Then Theorem 7.5 gives an algebra weight ω_1 and a nonzero continuous homomorphism of positive character $\phi : L^1(\omega_1) \to M(\omega_2)$. Now extend ϕ to a continuous homomorphism from L^1_{loc} to M_{loc} using Theorem 7.3. $\square$

8. Applications and open questions

The extension and restriction theorems for homomorphisms in Section 7 and the analogous theorems for semigroups and nets in Section 4 make it possible to prove results about L^1_{loc} from results about weighted convolution algebras and vice versa. We have already used this technique to obtain results, in particular Theorem 7.8, about L^1_{loc}. In this section we illustrate how to go in the other direction.

THEOREM 8.1. *Suppose that $\phi : L^1(\omega_1) \to M^1(\omega_2)$ is a continuous nonzero homomorphism of positive character, where ω_1 and ω_2 are algebra weights. Then there is a $0 \le c \le \infty$ for which $\phi(f) \in L^1(\omega_2)$ if and only if $\alpha(f) \ge c$.*

Notice that $c = 0$ means ϕ maps $L^1(\omega_1)$ into $L^1(\omega_2)$, which is usually the case we would prefer, and that $c = \infty$ means $\phi(f)$ is a function only for $f \equiv 0$.

Since, by Theorem 7.1, the map ϕ can be extended to be a continuous homomorphism from L^1_{loc} to M_{loc}, Theorem 8.1 is a consequence of the following lemma.

LEMMA 8.2. *Suppose that $\phi : L^1_{\mathrm{loc}} \to M_{\mathrm{loc}}$ is a continuous nonzero homomorphism. Then there is a $0 \le c \le \infty$ for which $\phi(f)$ belongs to L^1_{loc} if and only if $\alpha(f) \ge c$.*

PROOF. L^1_{loc} is a closed ideal in M_{loc}, so $\phi^{-1}(L^1_{\mathrm{loc}})$ is a closed ideal in L^1_{loc}. However, all closed ideals in L^1_{loc} are standard [10, Proposition 2.5 (a)]. Hence there is a $0 \le c \le \infty$ for which $\phi^{-1}(L^1_{\mathrm{loc}}) = L_c$. This proves the lemma and also proves Theorem 8.1. □

If the c in Theorem 8.1 is a positive finite number, then ϕ induces a homomorphism from $L^1[0, c)$ to $M_{\mathrm{loc}}/L^1_{\mathrm{loc}}$, and hence also induces a homomorphism from $L^1[0, c)$ to $M(\omega_2)/L^1(\omega_2)$. After moving to a quotient algebra, this continuously embeds $L^1[0, a)$ into $M(\omega_2)/L^1(\omega_2)$ for some $0 < a < c$. It is not clear that this can happen. This suggests the following question.

QUESTION 1. If $\phi : L^1(\omega_1) \to M(\omega_2)$ is a continuous homomorphism and if $\phi(f)$ is a function for some nonzero f in $L^1(\omega_1)$, must ϕ map $L^1(\omega_1)$ to $L^1(\omega_2)$? When ϕ has character 0, do we even have the result of Theorem 8.1?

Ever since homomorphisms of positive character were introduced in [13] (which drew heavily on arguments given for isomorphisms in [6]), these homomorphisms often turned out to be better behaved than homomorphisms of character 0. In this paper we saw, in Theorem 7.1, that these are precisely the homomorphisms that can be extended to L^1_{loc}. Other results, like Theorem 8.1, are known for all homomorphisms of positive character, but only for some homomorphisms of character 0. Below are some such properties from earlier papers. In the earlier papers the results are often given for homomorphisms from $L^1(\omega_1)$ to $L^1(\omega_2)$ and not also for those to $M(\omega_2)$. The more general case is proved similarly. The proofs in [13] work only for radical weights, but the simpler proofs in [15] work in general.

PROPOSITION 8.3. *Suppose that ω_1 and ω_2 are algebra weights and that $\phi : L^1(\omega_1) \to M(\omega_2)$ is a continuous nonzero homomorphism of positive character A. Then we have*

(a) [13, Theorem 4.9 (a)] *and* [15, Theorem 5.8] $\alpha(\phi(\mu)) = A\alpha(\mu)$.
(b) [13, page 613] *and* [15, page 188] ϕ *is one-one.*

We now know that (a) and (b) are consequences of the analogous results, Theorems 6.1 and 6.6, for L^1_{loc}. We only know Proposition 8.3 (a) and (b) for some homomorphisms of character 0, which suggests the following question (and its analogue for $\phi : L^1(\omega_1) \to M(\omega_2)$).

QUESTION 2. Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous homomorphism of character 0. Must ϕ satisfy conditions (a) or (b) of Proposition 8.3?

For ϕ of character 0, the analogue of Proposition 8.3 (a) would say $\alpha(\phi(f)) = 0$ for all nonzero f in $L^1(\omega_1)$. We do not know, in the characteristic 0 case, if this is true when $\alpha(f) = 0$, as in Lemma 6.2. A useful special case is [13, Lemma 4.5].

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