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THE COMMUTATOR SUBGROUP AND SCHUR MULTIPLIER OF A PAIR OF FINITE p-GROUPS

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Abstract

Let (M, G) be a pair of groups, in which M is a normal subgroup of G such that G/M and $M/Z(M, G)$ are of orders p^m and p^n, respectively. In 1998, Ellis proved that the commutator subgroup $[M, G]$ has order at most $p^{n(n+2m-1)/2}$.

In the present paper by assuming $|[M, G]| = p^{n(n+2m-1)/2}$, we determine the pair (M, G). An upper bound is obtained for the Schur multiplier of the pair (M, G), which generalizes the work of Green (1956).

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1. Introduction

Let (M, G) be a pair of groups such that M is a normal subgroup of G and N any other group. We recall from [5] that a *relative central extension* of the pair (M, G) is a group homomorphism $\sigma : N \to G$, together with an action of G on N (denoted by n^g, for all $n \in N$ and $g \in G$), such that the following conditions are satisfied:

(i) $\sigma(N) = M$;
(ii) $\sigma(n^g) = g^{-1}\sigma(n)g$, for all $g \in G$ and $n \in N$;
(iii) $n^{\sigma(n_1)} = n_1^{-1}nn_1$, for all $n, n_1 \in N$;
(iv) G acts trivially on $\ker\sigma$.

Taking $N = M$, clearly the inclusion map $i : M \to G$, acting by conjugation, is a simple example of a relative central extension of the pair (M, G).

Now for the given relative central extension σ, we define *G-commutator* and *G-central subgroups* of N, respectively, as follows

$$[N, G] = \langle [n, g] = n^{-1}n^{g} \mid n \in N, g \in G \rangle,$$
$$Z(N, G) = \{n \in N \mid n^{g} = n, \text{ for all } g \in G\}.$$

In special case $\sigma = i$, $[M, G]$ and $Z(M, G)$ are the commutator subgroup and the centralizer of G in M, respectively. In this case, we define $Z_2(M, G)$ to be the preimage in M of $Z(M/Z(M, G), G/Z(M, G))$, or

$$\frac{Z_2(M, G)}{Z(M, G)} = Z\left(\frac{M}{Z(M, G)}, \frac{G}{Z(M, G)}\right),$$

and inductively obtain the upper central series of the pair (M, G).

The pair (M, G) is said to be *capable* if it admits a relative central extension σ such that $\ker \sigma = Z(N, G)$ (see also [2]). One can easily see that this gives the usual notion of a capable group G [2], when the pair (G, G) is capable in the above sense.

We call a pair of finite p-groups (M, G) an *extra-special*, when $Z(M, G)$ and $[M, G]$ are the same subgroups of order p.

Ellis [3] defined the *Schur multiplier* of the *pair* (M, G) to be the abelian group $\mathscr{M}(M, G)$ appearing in the following natural exact sequence

$$H_3(G) \to H_3(G/M) \to \mathscr{M}(M, G) \to \mathscr{M}(G) \xrightarrow{\mu} \mathscr{M}(G/M)$$
$$\to M/[M, G] \to (G)^{ab} \to (G/M)^{ab} \to 0,$$

in which $\mathscr{M}(\cdot, \cdot)$ and $H_3(\cdot)$ are the Schur multiplier and the third homology of a group with integer coefficients, respectively. He also proved that if the factor groups G/M and $M/Z(M, G)$ are both finite of orders p^m and p^n, respectively, then the commutator subgroup $[M, G]$ is of order at most $p^{n(n+2m-1)/2}$. In this situation, the result of Wiegold in [8] is obtained, when $m = 0$. Now by the above discussion we state our first result, which generalizes the work of Berkovich [1].

THEOREM A. *Let (M, G) be a pair of finite p-groups with G/M and $M/Z(M, G)$ of orders p^m and p^n, respectively. If $|[M, G]| = p^{n(n+2m-1)/2}$, then either $M/Z(M, G)$ is an elementary abelian p-group or the pair $(M/Z(M, G), G/Z(M, G))$ is an extra-special pair of finite p-groups.*

Green, in [4], shows that if G is a group of order p^n, then its Schur multiplier is of order at most $p^{n(n-1)/2}$. The following theorem gives a similar result for the Schur multiplier of a pair of finite p-groups. Also, under some conditions we characterize the groups G, when the order of $\mathscr{M}(M, G)$ is either $p^{n(n+2m-1)/2}$ or $p^{n(n+2m-1)/2-1}$.

THEOREM B. *Let (M, G) be a pair of finite p-groups and N be the complement of M in G. Assume M and N are of orders p^n and p^m, respectively, then the following statements hold*:

(i) $|\mathcal{M}(M, G)| \leq p^{n(n+2m-1)/2}$;

(ii) *if G is abelian, N is elementary abelian, and* $|\mathcal{M}(M, G)| = p^{n(n+2m-1)/2}$, *then G is elementary abelian*;

(iii) *if the pair (M, G) is non-capable, and* $|\mathcal{M}(M, G)| = p^{n(n+2m-1)/2-1}$, *then* $G \cong \mathbb{Z}_{p^2}$.

2. Proof of theorems

Let (M, G) be a pair of finite p-groups with $|G/M| = p^m$ and $|M/Z(M, G)| = p^n$. It is easily seen that for any element $z \in Z_2(M, G) \backslash Z(M, G)$, the commutator map $\varphi : G \to [G, z]$ given by $\varphi(x) = [x, z]$ is an epimorphism. We note that $\mathrm{Im}\,\varphi \leq [M, G] \cap Z(M, G)$ and $Z(M, G) \leq \ker\varphi = C_G(z)$. Clearly $[M, G] \leq C_G(z)$, as $[G, z] \cong G/C_G(z)$. Consider two non-negative integers $\mu(z)$ and $\nu(z)$ such that

$$p^{\mu(z)} = |[G, z]|, \qquad p^{\nu(z)} = \left|\frac{G/[G, z]}{Z(M/[G, z], G/[G, z])}\right|.$$

Since $\ker\varphi = C_G(z) \supseteq \langle z, Z(M, G)\rangle \supset Z(M, G)$ and

$$z[G, z] \in Z(M/[G, z], G/[G, z]),$$

it follows that

(1) $$\mu(z) \leq m + n - 1 \quad \text{and} \quad \nu(z) \leq m + n - 1.$$

The following lemma shortens the proof of Theorem A.

LEMMA 2.1. (a) *Under the above assumptions and notation,*

$$|[M, G]| \leq p^{\{\nu(z)(\nu(z)-1)-m(m-1)\}/2+\mu(z)} \leq p^{n(n+2m-1)/2},$$

for all $z \in Z_2(M, G) \backslash Z(M, G)$.

(b) *Suppose for some non-negative integer s,* $|[M, G]| = p^{n(n+2m-1)/2-s}$, *then the following hold*:

(i) $\big|[M/Z(M, G), G/Z(M, G)]\big| \leq p^{s+1}$. *If* $\big|[M/Z(M, G), G/Z(M, G)]\big| = p^{s+1-k}$, *for some* $0 \leq k \leq s + 1$, *then* $\exp(Z_2(M, G)/Z(M, G)) \leq p^{k+1}$ *and* $\mu(z) \leq m + n - 1 - s + k$.

(ii) *If* $\exp(Z_2(M, G)/Z(M, G)) \geq p^k$, *then* $m + n \leq s/(k - 1) + k/2$.

PROOF. (a) Clearly $|G/Z(M,G)| = p^{n+m}$. By [9, Lemma 1], the inequality holds for $m = 0$ and $n \geq 1$. The case $m = 1$ and $n = 0$ is impossible, and hence one may assume that $n + m > 1$ and $m \neq 0$. Clearly for each $z \in Z_2(M,G) \backslash Z(M,G)$, it implies that $1 < |[G,z]| \leq |[M,G]|$, so using induction on $m + n$, we obtain

$$\left|\left[\frac{M}{[G,z]}, \frac{G}{[G,z]}\right]\right| = \left|\frac{[M,G]}{[G,z]}\right| \leq p^{\{\nu(z)(\nu(z)-1)-m(m-1\}/2}.$$

Thus

$$\begin{aligned} |[M,G]| = \left|\frac{[M,G]}{[G,z]}\right| |[G,z]| &\leq p^{\{\nu(z)(\nu(z)-1)-m(m-1\}/2+\mu(z)} \\ &\leq p^{\{(m+n-1)(m+n-2)-m(m-1)\}/2+(m+n-1)} \\ &= p^{n(n+2m-1)/2}. \end{aligned}$$

(b) By the assumptions and part (a), we have

$$\begin{aligned} p^{n(n+2m-1)/2-s} &\leq p^{\{\nu(z)(\nu(z)-1)-m(m-1)\}/2+\mu(z)} \\ &\leq p^{\{(m+n-1)(m+n-2)-m(m-1)\}/2+\mu(z)}, \end{aligned}$$

and so $\mu(z) \geq m + n - 1 - s$. Now, since $[M,G]Z(M,G)$ is a subgroup of $C_G(z)$, it implies that

$$[G : [M,G]Z(M,G)] \geq [G : C_G(z)] = p^{\mu(z)} \geq p^{m+n-1-s}.$$

The last inequality implies that

$$\left|\left[\frac{M}{Z(M,G)}, \frac{G}{Z(M,G)}\right]\right| \leq \frac{|G/Z(M,G)|}{p^{m+n-1-s}} = p^{s+1}.$$

Now, assuming that $|[M/Z(M,G), G/Z(M,G)| = p^{s+1-k}$, for some non-negative integer k, then

$$\begin{aligned} p^{\mu(z)} &\leq [G : [M,G]Z(M,G)] \\ &= \left[\frac{G}{Z(M,G)} : \left[\frac{M}{Z(M,G)}, \frac{G}{Z(M,G)}\right]\right] = p^{m+n-1-s+k}, \end{aligned}$$

and hence $\mu(z) \leq m + n - 1 - s + k$.

If $\exp(Z_2(M,G)/Z(M,G)) > p^{k+1}$, then there exists some $z \in Z_2(M,G)$ such that $z^{p^{k+1}} \notin Z(M,G)$. Thus

$$z[G,z] \in Z\left(\frac{M}{[G,z]}, \frac{G}{[G,z]}\right) \setminus \frac{Z(M,G)}{[G,z]},$$

which implies that

$$\left[Z\left(\frac{M}{[G,z]}, \frac{G}{[G,z]}\right) : \frac{Z(M,G)}{[G,z]} \right] \geq p^{k+2}.$$

Hence

$$p^{\nu(z)} = \frac{[G/[G,z] : Z(M,G)/[G,z]]}{[Z(M/[G,z], G/[G,z]) : Z(M,G)/[G,z]]} \leq \frac{p^{m+n}}{p^{k+2}} = p^{m+n-k-2},$$

and so $\nu(z) \leq m+n-k-2$. Hence using the hypothesis and part (a) we must have

$$\begin{aligned} n(n+2m-1)/2 - s \leq [(m+n-k-2)(m+n-k-3) - m(m-1)]/2 \\ + m + n - s - 1 + k \end{aligned}$$

or

$$2(k+1)(m+n) \leq k^2 + 7k + 4.$$

Therefore we have $m+n \leq k+2$ and so

$$p^{k+2} \leq \exp\left(\frac{Z_2(M,G)}{Z(M,G)}\right) \leq \left|\frac{M}{Z(M,G)}\right| \leq \left|\frac{G}{Z(M,G)}\right| \leq p^{m+n} \leq p^{k+2}.$$

This gives $M = G$, which is a contradiction and proves (i).

Now, to prove (ii) we use the assumption that there exists $z \in Z_2(M,G) \backslash Z(M,G)$ such that $|zZ(M,G)| \geq p^k$. Then $|C_G(z)| \geq |\langle z, Z(M,G)\rangle| \geq p^k |Z(M,G)|$, and hence $|[G,z]| \leq p^{m+n-k}$ which implies $\mu(z) \leq m+n-k$. With a similar argument to (i), we obtain $\nu(z) \leq m+n-k$. By part (a) we have

$$\begin{aligned} n(n+2m-1)/2 - s \leq [(m+n-k)(m+n-k-1) - m(m-1)]/2 \\ + m + n - k, \end{aligned}$$

and so $m+n \leq s/(k-1) + k/2$. □

Now we are ready to prove Theorem A.

PROOF OF THEOREM A. By applying Lemma 2.1 (b) in the case $s = 0$, we have

$$\left|\left[\frac{M}{Z(M,G)}, \frac{G}{Z(M,G)}\right]\right| \leq p.$$

Now consider two cases:

First assume $M/Z(M,G) = Z(M/Z(M,G), G/Z(M,G))$. Then $M/Z(M,G)$ is abelian and by Lemma 2.1 (a), $\exp(M/Z(M,G)) \leq p^2$. If the latter exponent is p^2, then by Lemma 2.1 (b), $m+n \leq 1$ in which case $M/Z(M,G)$ is of order at most p. When the exponent is p, then the factor group is elementary abelian p-group.

In the second case, assume

$$Z\left(\frac{M}{Z(M,G)}, \frac{G}{Z(M,G)}\right) \subset \frac{M}{Z(M,G)}.$$

Then by Lemma 2.1 (b),

$$\exp(Z_2(M,G)/Z(M,G)) = p.$$

Let $Z_2(M,G)/Z(M,G)$ have two distinct subgroups of orders p. Then there exist elements $y_0, z_0 \in Z_2(M,G)\backslash Z(M,G)$ such that

$$|\langle y_0 Z(M,G)\rangle| = |\langle z_0 Z(M,G)\rangle| = p$$

and

$$\langle y_0 Z(M,G)\rangle \cap \langle z_0 Z(M,G)\rangle = \langle Z(M,G)\rangle.$$

By Lemma 2.1 (b), for each $x_0 \in Z_2(M,G)\backslash Z(M,G)$, we have $\mu(x_0) = m+n-1$. Hence $G/C_G(y_0)$ and $G/C_G(z_0)$ are abelian groups of orders p^{m+n-1}, and so

$$[M,G] \leq C_G(y_0) \cap C_G(z_0) = Z(M,G),$$

which implies that

$$\left|\left[\frac{M}{Z(M,G)}, \frac{G}{Z(M,G)}\right]\right| = 1.$$

This is a contradiction and hence $Z_2(M,G)/Z(M,G)$ is an abelain group of order p. On the other hand, $[M/Z(M,G), G/Z(M,G)]$ is a subgroup of $Z_2(M,G)/Z(M,G)$ of order p, and so we must have

$$\frac{Z_2(M,G)}{Z(M,G)} = \left[\frac{M}{Z(M,G)}, \frac{G}{Z(M,G)}\right].$$

Thus $(M/Z(M,G), G/Z(M,G))$ is an extra-special pair of p-groups. □

Using Theorem A, we obtain the following corollary which is of interest in its own right.

COROLLARY 2.2. *Let (M,G) be a pair of finite p-groups with $|G/M| = p^m$, $|M/Z(M,G)| = p^n$, and $|[M,G]| = p^{n(n+2m-1)/2-s}$ for some $s \geq 0$. If there is a $z_0 \in Z_2(M,G)\backslash Z(M,G)$ such that $\mu(z_0) = m+n-1-s$, then $\nu(z_0) = m+n-1$ and*

$$\frac{M/[G,z_0]}{Z(M/[G,z_0], G/[G,z_0])}$$

is elementary abelian p-group of order p^{m+n-1}, or

$$\left(\frac{M/[G,z_0]}{Z(M/[G,z_0],G/[G,z_0])},\frac{G/[G,z_0]}{Z(M/[G,z_0],G/[G,z_0])}\right)$$

is an extra-special pair of p-groups.

PROOF. Using equation (1) and Lemma 2.1 (a), we have

$$\begin{aligned} n(n+2m-1)/2-s &\leq [\nu(z_0)(\nu(z_0)-1)-m(m-1)]/2+\mu(z_0) \\ &\leq [(m+n-1)(m+n-2)-m(m-1)]/2 \\ &\quad +m+n-1-s, \end{aligned}$$

which implies that $\nu(z_0)=m+n-1$.

Hence

$$\left|\frac{M/[G,z_0]}{Z(M/[G,z_0],G/[G,z_0])}\right|=p^{n-1},$$

and also

$$\left|\left[\frac{M}{[G,z_0]},\frac{G}{[G,z_0]}\right]\right|=\left|\frac{[M,G]}{[G,z_0]}\right|=p^{n(n+2m-1)/2-s-m-n+s+1}=p^{(n-1)(n+2m-2)/2}.$$

Then the result follows from Theorem A. □

To prove Theorem B, we recall the concept of covering pair from [3].

The relative central extension $\sigma: M^*\to G$ is called a *covering pair* of the pair of finite groups (M,G) when the following conditions are satisfied:

(i) $\ker\sigma\subseteq Z(M^*,G)\cap[M^*,G]$;
(ii) $\ker\sigma\cong\mathscr{M}(M,G)$;
(iii) $M\cong M^*/\ker\sigma$.

If $\sigma: G^*\to G$ is a covering pair of the pair (G,G), then G^* is the usual covering group of G, which was introduced by Schur [7].

In [3], Ellis proved that any finite pair of groups admits a covering pair. The first two authors, under certain conditions in [6], showed the existence of a covering pair for an arbitrary pair of groups.

PROOF OF THEOREM B. Let $\sigma: M^*\to G$ together with an action of G on M^* be a covering pair of (M,G). We define a homomorphism $\psi: N\to \mathrm{Aut}(M^*)$ given by $\psi(n)=\psi_n$, for all $n\in N$, where $\psi_n: M^*\to M^*$, $m\mapsto m^n$ is an automorphism, in which m^n is induced by the action of G on M^*. We form the semidirect product of M^* by N and denote it by $H=M^*N$. Then one may easily check that the subgroup $[M^*,G]$ and $Z(M^*,G)$ are contained in $[M^*,H]$ and $Z(M^*,H)$, respectively. If

$\delta : H \to G$ is the mapping given by $\delta(mn) = \sigma(m)n$, for all $m \in M^*$ and $n \in N$, then it is easily seen that δ is an epimorphism with $\ker\delta = \ker\sigma$.

(i) Since $|H/M^*| = p^m$ and $|M^*/Z(M^*, H)| \le p^n$, then by Lemma 2.1 (a),

$$|\mathscr{M}(M, G)| \le |[M^*, H]| \le p^{n(n+2m-1)/2}.$$

(ii) By [1, Theorem 2.1], $|\mathscr{M}(N)| = p^{m(m-1)/2}$. Since the exact sequence

$$1 \to M \to G \to N \to 1$$

splits, it follows easily that $\mathscr{M}(G) = \mathscr{M}(M, G) \oplus \mathscr{M}(N)$. Hence $|\mathscr{M}(G)| = p^{(n+m)(n+m-1)/2}$ and so again by [1, Theorem 2.1], G is an elementary abelian p-group.

(iii) By assumption, $\ker\sigma$ is a proper subgroup of $Z(M^*, H)$, so

$$|M^*/Z(M^*, H)| \le p^{n-1}.$$

Hence by Lemma 2.1 (a), $|[M^*, H]| \le p^{(n-1)(2m+n-2)/2}$. On the other hand, we have $\mathscr{M}(M, G) \cong \ker\sigma \le [M^*, H]$. Therefore

$$n(2m + n - 1)/2 - 1 \le (n - 1)(2m + n - 2)/2$$

and so $m + n \le 2$. But since the case $m + n = 1$ is impossible, it implies $m + n = 2$. In the latter case, we must have $n = 2$ and $m = 0$. Now, if $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, then $|\mathscr{M}(M, G)| = |\mathscr{M}(G)| = p$, which is a contradiction. Hence $G \cong \mathbb{Z}_{p^2}$, which completes the proof. $\square$

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