

A PROPERTY OF THE COMPLEX SEMIGROUP ALGEBRA OF A FREE MONOID

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Abstract

It is shown that the complex semigroup algebra of a free monoid of rank at least two is $*$ -primitive, where $*$ denotes the involution on the algebra induced by word-reversal on the monoid.

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Let A be an algebra over the complex field \mathbb{C} that admits an involution $*$; thus $*$ is a mapping $A \rightarrow A$ such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$

$$(a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*, \quad a^{**} = a, \quad (\lambda a)^* = \bar{\lambda}a^*,$$

where $\bar{\lambda}$ denotes the complex conjugate of λ . A right module V for A is termed a $*$ -module if and only if it admits an inner product $\langle | \rangle$ such that

$$\langle ua|v \rangle = \langle u|va^* \rangle \quad \text{for all } u, v \in V \text{ and } a \in A.$$

We say that A is $*$ -primitive if and only if it has a faithful irreducible $*$ -module.

The complex semigroup algebra of a semigroup S is denoted by $\mathbb{C}[S]$. For a nonempty set X , the free monoid and the free group on X are denoted, respectively, by M_X and G_X . Note that $\mathbb{C}[M_X]$ is the free complex algebra-with-unity on X . It is well known and easy to see that each of the algebras $\mathbb{C}[M_X]$ and $\mathbb{C}[G_X]$ possesses an involution. Let $*$ denote the involution on $\mathbb{C}[M_X]$ defined by

$$\left(\sum_{i=1}^n \alpha_i y_i \right)^* := \sum_{i=1}^n \bar{\alpha}_i \overleftarrow{y_i} \quad \text{for } \alpha_i \in \mathbb{C}, y_i \in M_X,$$

where $\overleftarrow{y_i}$ denotes the reverse of the word y_i , and let \dagger denote the involution on $\mathbb{C}[G_X]$ defined by

$$\left(\sum_{i=1}^n \alpha_i g_i \right)^\dagger := \sum_{i=1}^n \bar{\alpha}_i g_i^{-1} \quad \text{for } \alpha_i \in \mathbb{C}, g_i \in G_X.$$

Now suppose that X has at least 2 elements. It was shown by Formanek [4] that $\mathbb{C}[G_X]$ is primitive (that is, has a faithful irreducible right module); and his argument can be adapted to also show that $\mathbb{C}[M_X]$ is primitive (see [8, Chapter 9, Ex. 17]). Subsequently, explicit constructions for faithful irreducible right modules for $\mathbb{C}[M_X]$ and $\mathbb{C}[G_X]$ were provided by McGregor ([7] and [6]); and alternative constructions, without cardinality restrictions, appeared in [1] and [2]. As was pointed out by Irving [5], the module constructed for $\mathbb{C}[G_X]$ in [6] is in fact a \dagger -module; thus $\mathbb{C}[G_X]$ is \dagger -primitive. The purpose of the present paper is to show that $\mathbb{C}[M_X]$ is $*$ -primitive. This does not appear to follow from the construction in [7]. To obtain the result, we adapt the procedure that establishes the \dagger -primitivity of $\mathbb{C}[G_X]$.

The symbols \mathbb{N} and \mathbb{Z} denote, respectively, the sets of all positive integers and all integers and $|S|$ denotes the cardinal of a set S . Let X be a set with $|X| \geq 2$ and let s, t be distinct elements of X . The identity of G_X (the empty word) is denoted by 1 and the set $\{x^{-1} : x \in X\}$ by X^{-1} . If $g \in G_X \setminus \{1\}$ has reduced form $g = g_1 g_2 \cdots g_n$, where $g_1, g_2, \dots, g_n \in X \cup X^{-1}$, then we write

$$\begin{aligned} l(g) &:= n, & g^\# &:= g_1^{-1} g_2^{-1} \cdots g_n^{-1}, \\ g^\Omega &:= g_n, & g^\flat &:= g_1 g_2 \cdots g_{n-1} \quad (= 1 \text{ if } n = 1). \end{aligned}$$

We also take $l(1) = 0$. Next, we write

$$L := \left\{ g \in G_X \mid \begin{array}{l} g \text{ has reduced form } s^k g_1 g_2 \cdots g_n \text{ for} \\ k \in \mathbb{Z} \setminus \{0\}, 0 \leq n \leq |k|, g_i \in X \cup X^{-1} \end{array} \right\} \cup \{1\}$$

and $E := \{g \in G_X : g \notin L \text{ and } g^\flat \in L\}$. As in [6], we use these sets to define subsets $\mathcal{L}, \mathcal{E}, \mathcal{U}^+, \mathcal{U}^-, \mathcal{U}$, and \mathcal{B} of $G_X \times \mathbb{Z}$ by $\mathcal{L} := L \times \{0\}$, $\mathcal{E} := E \times \{0\}$,

$$\begin{aligned} \mathcal{U}^+ &:= \{(w, n) : w \in E, w^\Omega \in X \text{ and } n \in \mathbb{N}\}, \\ \mathcal{U}^- &:= \{(w, -n) : w \in E, w^\Omega \in X^{-1} \text{ and } n \in \mathbb{N}\}, \end{aligned}$$

$\mathcal{U} := \mathcal{U}^+ \cup \mathcal{U}^-$, and $\mathcal{B} := \mathcal{L} \cup \mathcal{E} \cup \mathcal{U}$. We also define a subset \mathcal{U}^* of \mathcal{U} by

$$\mathcal{U}^* := \{(t, 3^n) : n \in \mathbb{N} \cup \{0\}\}.$$

In [6], \mathcal{U}^* is taken to be $\{(t, 2^n) : n \in \mathbb{N}\}$, but this change does not affect the validity of the construction.

It may be verified that \mathcal{B} has cardinal $\max\{|X|, \aleph_0\}$. Let V be the complex vector space consisting of all mappings $\mathcal{B} \rightarrow \mathbb{C}$ of finite support, so we may write a typical element of V in the form $\sum_{i=1}^n \alpha_i e_i$ for some $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$ and $e_i \in \mathcal{B}$. Again, following [6], we define a right action of $\mathbb{C}[G_X]$ on V . First, we define $ex \in V$ for $e \in \mathcal{B}$ and $x \in X$ by the rules below:

$$\begin{aligned} & \text{for all } (w, 0) \in \mathcal{L}, \quad (w, 0)x = (wx, 0) \in \mathcal{L} \cup \mathcal{E}, \\ & \text{for all } (w, 0) \in \mathcal{E}, \quad (w, 0)x = \begin{cases} (w, 1) \in \mathcal{U}^+ & \text{if } w^\Omega \in X, \\ (w^\flat, 0) \in \mathcal{L} & \text{if } w^\Omega = x^{-1}, \\ (w^\sharp, 0) \in \mathcal{E} & \text{otherwise,} \end{cases} \\ & \text{for all } (w, k) \in \mathcal{U}, \quad (w, k)x = \begin{cases} -(w, k+1) & \text{if } x = s \text{ and } (w, k) \in \mathcal{U}^*, \\ (w, k+1) & \text{otherwise.} \end{cases} \end{aligned}$$

It can be shown that for all $x \in X \setminus \{s\}$ the mapping $\mathcal{B} \rightarrow \mathcal{B}$, $e \mapsto ex$ is a permutation. Thus we may extend it by linearity to an invertible mapping $V \rightarrow V$, $v \mapsto vx$. Although the rule $e \mapsto es$ for $e \in \mathcal{B}$ does not give a permutation of \mathcal{B} , it also extends to an invertible mapping $V \rightarrow V$, $v \mapsto vs$. For all $v \in V$ we take $v1 := v$. Next, we define $vx^{-1} \in V$ for all $v \in V$ and all $x \in X$ by $vx^{-1} = w$, where $wx = v$. This enables us to define a right action of G_X on V and hence a right action of $\mathbb{C}[G_X]$ on V .

The first lemma states that, with respect to this action, V is a \dagger -module.

LEMMA 1 (Irving [5]). *Let $\langle \cdot | \cdot \rangle$ be the inner product on V defined by*

$$\text{for all } e, f \in \mathcal{B}, \quad \langle e | f \rangle = \begin{cases} 1 & \text{if } e = f, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle ua | v \rangle = \langle u | va^\dagger \rangle$ for all $u, v \in V$ and $a \in \mathbb{C}[G_X]$.

We now gather together some further properties of V for ease of reference. These properties are straightforward consequences of the action on V and are mostly stated in [6, Lemma 1].

LEMMA 2. (i) $et \in \mathcal{U}^+$ for all $e \in \mathcal{U}^+$, $et^{-1} \in \mathcal{U}^-$ for all $e \in \mathcal{U}^-$;
(ii) for all $e \in \mathcal{B}$, there exists $n \in \mathbb{N}$ such that $et^n \in \mathcal{U}^+$ and $et^{-n} \in \mathcal{U}^-$;
(iii) $(s^r, 0)g \in \mathcal{L}$ for all $r \in \mathbb{N}$ and $g \in G_X$ with $l(g) \leq r$;
(iv) for all $r \in \mathbb{N}$ and $g, g' \in G_X$ with $l(g), l(g') \leq r$, $(s^r, 0)g = (s^r, 0)g'$ implies $g = g'$.

Next, as in the proof of [3, Theorem 1.1], we define a homomorphism $\theta : \mathbb{C}[M_X] \rightarrow \mathbb{C}[G_X]$ by $\theta(x) := x + x^{-1}$ for all $x \in X$. Any mapping $X \rightarrow \mathbb{C}[G_X]$ extends

uniquely to a monoid homomorphism $M_X \rightarrow (\mathbb{C}[G_X], \cdot)$ and hence to an algebra homomorphism $\mathbb{C}[M_X] \rightarrow \mathbb{C}[G_X]$. The lemma below lists some properties of θ .

LEMMA 3. (i) θ is an injective homomorphism;

(ii) for each $n \in \mathbb{N}$, there exists a polynomial f_n over \mathbb{Z} of degree n such that, for all $x \in X$, $x^n + x^{-n} = f_n(\theta(x))$;

(iii) for all $a \in \mathbb{C}[M_X]$, $(\theta(a))^\dagger = \theta(a^*)$.

PROOF. (i) We may regard M_X as a submonoid of G_X . Let $a \in \mathbb{C}[M_X] \setminus \{0\}$. Consider an element w of $\text{supp}(a)$ with $l(w)$ maximal. Then $w \in \text{supp}(\theta(a))$, which shows that $\theta(a) \neq 0$. Hence θ is injective.

(ii) This can be established by induction. In fact, f_n is closely related to the n th Chebychev polynomial of the first type.

(iii) For all $x \in X$, $(\theta(x))^\dagger = x + x^{-1} = \theta(x)$ and so, for all $y \in M_X$, $(\theta(y))^\dagger = \theta(\overleftarrow{y})$. Hence, for all $a \in \mathbb{C}[M_X]$, $(\theta(a))^\dagger = \theta(a^*)$. \square

Denote the element $(t, 1)$ of \mathcal{B} by e_1 and define $W \subseteq V$ by

$$W := \{e_1\theta(a) : a \in \mathbb{C}[M_X]\}.$$

Then W is a nonzero subspace of V . Next, we define $\circ : W \times \mathbb{C}[M_X] \rightarrow W$ by $w \circ a = w\theta(a)$ for $w \in W$, $a \in \mathbb{C}[M_X]$. It is straightforward to see that \circ is a right action of $\mathbb{C}[M_X]$ on W . We now show that W is faithful and irreducible under this action.

LEMMA 4. W is a faithful module for $\mathbb{C}[M_X]$.

PROOF. Let $a \in \mathbb{C}[M_X] \setminus \{0\}$. Then, by Lemma 3 (i), $\theta(a) \in \mathbb{C}[G_X] \setminus \{0\}$. Thus $\theta(a) = \sum_{i=1}^n \alpha_i g_i$ for some $n \in \mathbb{N}$, some distinct elements $g_i \in G_X$, and some coefficients α_i , not all zero. Take

$$r := \max\{l(g_i) : i = 1, \dots, n\} + 5$$

and write

$$w := e_1(t^2 + t^{-2})(s^r + s^{-r}).$$

Since $(t^2 + t^{-2})(s^r + s^{-r}) = \theta(f_2(t)f_r(s))$, by Lemma 3 (ii), we have that $w \in W$. The action of t and of s on certain elements of \mathcal{B} can be represented diagrammatically as

$$\begin{aligned} t : \dots &\rightarrow (t^{-1}, -1) \rightarrow (t^{-1}, 0) \rightarrow (1, 0) \rightarrow (t, 0) \rightarrow (t, 1) \rightarrow (t, 2) \rightarrow \dots, \\ s : \dots &\rightarrow (t^{-1}, -1) \rightarrow (t^{-1}, 0) \rightarrow (t, 0) \rightarrow (t, 1) \rightarrow -(t, 2) \rightarrow -(t, 3) \rightarrow \dots. \end{aligned}$$

Hence we have that

$$(1) \quad \begin{aligned} w &= [(t, 3) + (1, 0)](s^r + s^{-r}) \\ &= \pm(t, r+3) - (t^{-1}, -r+4) + (s^r, 0) + (s^{-r}, 0). \end{aligned}$$

From the choice of r ,

$$(2) \quad \pm(t, r+3)g_i \in \mathcal{U}^+, \quad (t^{-1}, -r+4)g_i \in \mathcal{U}^- \quad \text{for } i = 1, 2, \dots, n;$$

and

$$(3) \quad \{(s^{-r}, 0)g_i : i = 1, 2, \dots, n\} \cap \{(s^r, 0)g_i : i = 1, 2, \dots, n\} = \emptyset.$$

Now, by Lemma 2 (iv), since the g_i are distinct so are the elements $(s^r, 0)g_i$ for $i = 1, \dots, n$. However, by Lemma 2 (iii), these lie in \mathcal{L} . Hence, from (1)–(3), $w\theta(a) \neq 0$, that is, $w \circ a \neq 0$. Thus W is faithful. \square

LEMMA 5. *W is an irreducible module for $\mathbb{C}[M_X]$.*

PROOF. Take $\langle | \rangle$ to be the inner product on V defined as in Lemma 1. Let $w \in W \setminus \{0\}$. Then $w = e_1\theta(a)$ for some $a \in \mathbb{C}[M_X]$ and so $\langle e_1\theta(a) | e_1\theta(a) \rangle \neq 0$. However, by Lemma 1 and Lemma 3 (iii),

$$\langle e_1\theta(a) | e_1\theta(a) \rangle = \langle e_1 | e_1\theta(a)(\theta(a))^\dagger \rangle = \langle e_1 | w\theta(a^*) \rangle = \langle e_1 | w \circ a^* \rangle$$

and so the coefficient of e_1 in $w \circ a^*$ is nonzero. Thus we may write $w \circ a^* = \sum_{i=1}^n \alpha_i e_i$ for some $n \in \mathbb{N}$, some distinct $e_i \in \mathcal{B}$ with $e_1 = (t, 1)$, and some nonzero coefficients α_i for $i = 1, \dots, n$.

By Lemma 2 (i) and (ii), there exists $p \in \mathbb{N}$ such that

$$e_i t^p \in \mathcal{U}^+, \quad e_i t^{-p} \in \mathcal{U}^- \quad \text{for } i = 1, \dots, n.$$

These $2n$ elements are distinct. Write $(g_i, k_i) := e_i t^p$ for $i = 1, \dots, n$. In particular, $(g_1, k_1) = (t, 1)t^p = (t, p+1)$. Let $l \in \mathbb{N}$ be defined by

$$l := \max\{k_i : 1 \leq i \leq n \text{ and } g_i = t\}.$$

Choose $m \in \mathbb{N}$ such that $3^{m-1} > l$ and take $q := 3^m - l$. Then

$$(4) \quad e_i t^{p+q} = (g_i, k_i + q) = (g_i, 3^m - l + k_i) \quad \text{for } i = 1, \dots, n.$$

Let $j \in \{1, \dots, n\}$ be such that $g_j = t$ and $k_j = l$. Then, by (4), $e_j t^{p+q} = (t, 3^m)$ and so

$$(5) \quad e_j t^{p+q}(t-s) = 2(t, 3^m + 1), \quad e_j t^{p+q}(t^{-1} - s^{-1}) = 0.$$

We next show that

$$(6) \quad e_i t^{p+q}(t-s) = 0, \quad e_i t^{p+q}(t^{-1} - s^{-1}) = 0 \quad (i \neq j).$$

Let $i \in \{1, \dots, n\}$ with $i \neq j$. First, suppose that $g_i = t$. Then $k_i < l$ and so $3^m - l + k_i < 3^m$. Further, since $3^{m-1} > l$, we have that $3^m - l > 3^{m-1} + l$, and so $3^m - l + k_i > 3^{m-1} + 1$. Since $e_i t^{p+q} = (t, 3^m - l + k_i)$, by (4), it follows that (6) holds. Now suppose that $g_i \neq t$. Then, from (4), we see that (6) holds in this case also. Thus we have established (6). Since $e_i t^{-p} \in \mathcal{U}^-$,

$$(7) \quad e_i t^{-p-q}(t-s) = 0, \quad e_i t^{-p-q}(t^{-1} - s^{-1}) = 0 \quad \text{for } i = 1, \dots, n.$$

Write $u := t + t^{-1} - s - s^{-1}$. Then, by (5)–(7),

$$\begin{aligned} (w \circ a^*)(t^{p+q} + t^{-p-q})u &= (w \circ a^*)t^{p+q}(t-s) + (w \circ a^*)t^{p+q}(t^{-1} - s^{-1}) \\ &\quad + (w \circ a^*)t^{-p-q}(t-s) + (w \circ a^*)t^{-p-q}(t^{-1} - s^{-1}) \\ &= 2\alpha_j(t, 3^m + 1). \end{aligned}$$

Now write $r := 3^m - 1$. Then $(t, 3^m + 1)(t^r + t^{-r}) = (t, 2 \cdot 3^m) + (t, 2)$ and so

$$\begin{aligned} (t, 3^m + 1)(t^r + t^{-r})u &= (t, 2 \cdot 3^m)(t-s) + (t, 2 \cdot 3^m)(t^{-1} - s^{-1}) \\ &\quad + (t, 2)(t-s) + (t, 2)(t^{-1} - s^{-1}) \\ &= 2(t, 1) = 2e_1. \end{aligned}$$

Hence

$$(8) \quad (w \circ a^*)(t^{p+q} + t^{-p-q})u(t^r + t^{-r})u = 4\alpha_j e_1.$$

Let $b \in \mathbb{C}[M_X]$ be defined by $b := f_{p+q}(t)(t-s)f_r(t)(t-s)$, where f_{p+q} and f_r are the polynomials defined in Lemma 3 (ii). Then $\theta(b) = (t^{p+q} + t^{-p-q})u(t^r + t^{-r})u$ and so, from (8), $w \circ (a^*b) = (w \circ a^*)\theta(b) = 4\alpha_j e_1$. Since $\alpha_j \neq 0$, it follows that $w \circ \mathbb{C}[M_X] = W$. Thus W is irreducible. \square

The main result now follows.

THEOREM 6. *Let M_X denote the free monoid on a set X with at least two elements and let $*$ denote the involution on $\mathbb{C}[M_X]$ induced by word-reversal. Then $\mathbb{C}[M_X]$ is $*$ -primitive.*

PROOF. By Lemmas 4 and 5, W is a faithful irreducible module for $\mathbb{C}[M_X]$. Now, by Lemma 1, there exists an inner product $\langle | \rangle$ on V such that, for all $u, v \in V$ and

all $b \in \mathbb{C}[G_X]$, $\langle ub|v \rangle = \langle u|vb^\dagger \rangle$. Consider the restriction of this inner product to W . Then, for all $w_1, w_2 \in W$ and all $a \in \mathbb{C}[M_X]$,

$$\begin{aligned} \langle w_1 \circ a | w_2 \rangle &= \langle w_1 \theta(a) | w_2 \rangle = \langle w_1 | w_2 (\theta(a))^\dagger \rangle \\ &= \langle w_1 | w_2 \theta(a^*) \rangle, \quad \text{by Lemma 3 (iii),} \\ &= \langle w_1 | w_2 \circ a^* \rangle. \end{aligned}$$

Hence W is a $*$ -module and so W is $*$ -primitive. □

REMARK. The construction in [7] also shows that the Banach algebra $l^1(M_X)$ is primitive for the case $|X| \geq 2$. The question of whether $l^1(M_X)$ is $*$ -primitive in this case remains open.

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