

MODULES WHICH ARE INVARIANT UNDER MONOMORPHISMS OF THEIR INJECTIVE HULLS

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Abstract

In this paper certain injectivity conditions in terms of extensions of monomorphisms are considered. In particular, it is proved that a ring R is a quasi-Frobenius ring if and only if every monomorphism from any essential right ideal of R into $R_R^{(N)}$ can be extended to R_R . Also, known results on pseudo-injective modules are extended. Dinh raised the question if a pseudo-injective CS module is quasi-injective. The following results are obtained: M is quasi-injective if and only if M is pseudo-injective and M^2 is CS. Furthermore, if M is a direct sum of uniform modules, then M is quasi-injective if and only if M is pseudo-injective. As a consequence of this it is shown that over a right Noetherian ring R , quasi-injective modules are precisely pseudo-injective CS modules.

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1. Introduction

Throughout the paper rings are associative with identity and modules are unitary (right) modules. Let M and N be two right R -modules over a ring R . M is called (*pseudo*-) N -*injective* if, for any submodule A of N , every homomorphism (monomorphism) in $\text{Hom}_R(A, M)$ can be extended to an element of $\text{Hom}_R(N, M)$. M is called *quasi-injective* (*pseudo-injective*) if it is (*pseudo*-) M -*injective*. M and N are called relatively injective if M is N -*injective* and N is M -*injective*. A submodule K of M is said to be a complement in M of a submodule B if K is a maximal submodule among those that have zero intersection with B . Complement submodules of M coincide with the submodules of M which do not have any proper essential extension in M . Also, if A is a complement in M and B is a complement in A , then B is a complement in M .

A CS module is one in which complement submodules are direct summands. M is called a continuous module if it is a CS module and submodules of M isomorphic to direct summands of M are again direct summands. If M is continuous and A and B are two direct summands of M with $A \cap B = 0$, then $A \oplus B$ is also a direct summand of M . The hierarchy is as follows:

$$\text{Injective} \implies \text{quasi-injective} \implies \text{continuous} \implies \text{CS}.$$

For other properties of complements and CS/continuous modules and the proofs of the above mentioned properties, the reader is referred to [3] and [10].

In this paper, a weaker form of pseudo- N -injectivity is considered, and it is proved, in particular, that a ring R is quasi-Frobenius if and only if monomorphisms from essential right ideals of R into $R^{(\mathbb{N})}$ can be extended to R_R . Also it is shown that a module M is invariant under monomorphisms of its injective hull if and only if every monomorphism from any essential submodule of M can be extended to M . This extension property is used to characterize when semi-prime/right nonsingular rings are SI (see [6]).

Pseudo-injectivity has been studied by several authors such as Dinh, Jain, Singh, Teply, Tuganbaev and others (see [2, 8, 9, 13, 14, 15]). It was first introduced by Jain and Singh [8]. Teply [14] constructed examples of pseudo-injective modules which are not quasi-injective. In [2] Dinh raised the question if a pseudo-injective CS module is quasi-injective. He stated in [2] that the answer is affirmative if we assume further that M is nonsingular. In this paper we prove the following: M is quasi-injective if and only if M is pseudo-injective and M^2 is CS. Every uniform pseudo-injective module is quasi-injective. Consequently, over a right Noetherian ring R , quasi-injective modules are precisely pseudo-injective CS modules.

2. Essentially pseudo- N -injectivity

In this section we consider a weaker form of pseudo- N -injectivity.

DEFINITION 2.1. Let M and N be two modules. M is said to be *essentially pseudo- N -injective* if for any essential submodule A of N , any monomorphism $f : A \rightarrow M$ can be extended to some $g \in \text{Hom}(N, M)$. M is called *essentially pseudo-injective* if M is essentially pseudo- M -injective.

Obviously any pseudo- N -injective module is essentially pseudo- N -injective, but the converse is not true in general.

EXAMPLE 1. Let p be a prime. The \mathbb{Z} -module $\mathbb{Z}/p^2\mathbb{Z}$ is not pseudo- $(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z})$ -injective since the obvious isomorphism $\iota : p\mathbb{Z}/p^3\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z}$ can not be extended

to any element of $\text{Hom}(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z})$, but it is essentially pseudo- $(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z})$ -injective.

The following proposition provides a characterization of essentially pseudo- N -injectivity.

PROPOSITION 2.2. *Let M and N be two modules and $X = M \oplus N$. The following conditions are equivalent:*

- (i) M is essentially pseudo- N -injective.
- (ii) For any complement K in X of M with $K \cap N = 0$, $M \oplus K = X$.

PROOF. (i) \Rightarrow (ii) Let K be a complement in X of M with $K \cap N = 0$, and $\pi_M : M \oplus N \rightarrow M$ and $\pi_N : M \oplus N \rightarrow N$ be the obvious projections. Note that $M \oplus K = M \oplus \pi_N(K)$ so that $\pi_N(K)$ is essential in N .

Now define $\theta : \pi_N(K) \rightarrow \pi_M(K)$ as follows: For $k \in K$ with $k = m + n$ ($m \in M, n \in N$), $\theta(n) = m$. Then θ is a monomorphism by the $K \cap N = 0$ assumption. Hence θ can be extended to some $g : N \rightarrow M$, since M is essentially pseudo- N -injective. Now let $T = \{n + g(n) : n \in N\}$. It is easy to see that $M \oplus T = X$. Also, T contains K essentially by modularity. Since K is a complement, this implies $T = K$. Now the conclusion follows.

(ii) \Rightarrow (i) Assume (ii). Let A be an essential submodule of N and $f : A \rightarrow M$ be a monomorphism. Let $H = \{a - f(a) : a \in A\}$. Obviously, $H \cap N = 0$. Also note that $M \oplus H = M \oplus \pi_N(H) = M \oplus A$, which is essential in X . Let K be a complement in X of M containing H . By the previous argument and modularity H is essential in K , so that $K \cap N = 0$. By assumption we have $M \oplus K = X$. Now let $\phi : M \oplus K \rightarrow M$ be the obvious projection. Then the restriction $\phi|_N$ is the desired extension of f . The proof is now complete. \square

PROPOSITION 2.3. *If M is essentially pseudo- N -injective, every direct summand of M is essentially pseudo- N -injective.*

PROOF. Let $X = M \oplus N$ and assume $M = M_0 \oplus A$. Let K be a complement in $M_0 \oplus N$ of M_0 with $K \cap N = 0$. Then $M \oplus K$ is essential in X . Since K is a complement submodule, the preceding argument implies that K is also a complement in X of M . Now by Proposition 2.2 $M \oplus K = X$. Then $M_0 \oplus K = M_0 \oplus N$, which yields the conclusion again by Proposition 2.2. \square

The next example shows that essentially pseudo- N -injectivity is not inherited by direct sums.

EXAMPLE 2. Let F be a field and

$$R = \begin{pmatrix} F & F \oplus F \\ 0 & F \end{pmatrix}.$$

Consider the R -modules

$$N = \begin{pmatrix} F & F \oplus F \\ 0 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 \oplus F \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & F \oplus 0 \\ 0 & 0 \end{pmatrix}.$$

Then S_1 and S_2 are both essentially pseudo- N -injective. But since the identity map of $S_1 \oplus S_2$ obviously can not be extended to an element of $\text{Hom}(N, S_1 \oplus S_2)$, $S_1 \oplus S_2$ is not essentially pseudo- N -injective.

PROPOSITION 2.4. *Let M and N be two modules. Then the following conditions are equivalent:*

- (i) M is N -injective.
- (ii) M is essentially pseudo- N/L -injective for every submodule L of N .

PROOF. (i) \Rightarrow (ii) follows from [10, Proposition 1.3].

(ii) \Rightarrow (i) Assume M is essentially pseudo- N/L -injective for every submodule L of N . Let $X = M \oplus N$, $A \subseteq X$ with $A \cap M = 0$ and K be a complement in X of M containing A . Also let $T = K \cap N$. Since $(M \oplus K)/K$ is essential in X/K , then $(M \oplus K)/T$ is essential in X/T , and $K/T \cap N/T = 0$. Thus it is easy to see that K/T is a complement in X/T of $(M \oplus T)/T$. Now by assumption and Proposition 2.2 we have $(M \oplus T)/T \oplus K/T = X/T$. Hence $M \oplus K = X$. Then by [3, Lemma 7.5] M is N -injective. \square

COROLLARY 2.5. *M is injective if and only if M is essentially pseudo- N -injective for any cyclic module N .*

COROLLARY 2.6. *A nonsingular module M is injective if and only if it is essentially pseudo- N -injective for any nonsingular cyclic module N .*

PROOF. Let A be any cyclic module and B be an essential submodule of A . Let $f : B \rightarrow M$ be a monomorphism. Then A is obviously nonsingular, so that f can be extended to some $g : A \rightarrow M$ by assumption. Now the result follows by Corollary 2.5. \square

The following result generalizes [2, Theorem 2.2] and [9, Theorem 1].

THEOREM 2.7. *If $M \oplus N$ is essentially pseudo- N -injective, then M is N -injective.*

PROOF. Call $X = M \oplus N$. Let A and K be as in the proof of Proposition 2.4. Let $\pi : M \oplus N \rightarrow N$ be the obvious projection. Then $M \oplus K = M \oplus \pi(K)$ and thus $\pi(K)$ essential in N . Note that $K \cong \pi(K)$. Pick any isomorphism $f : \pi(K) \rightarrow K$. By assumption f can be extended to some monomorphism $g : N \rightarrow X$. Then $g(\pi(K)) = K$ is essential in $g(N)$. But since K is a complement in X , we must have $K = g(N)$, whence $\pi(K) = N$. Thus $M \oplus K = X$. Now the result follows by [3, Lemma 7.5]. \square

COROLLARY 2.8. *M is quasi-injective if and only if M^2 is essentially pseudo- M -injective.*

Osofsky proved in [12] that a ring R is semisimple Artinian if and only if every cyclic right (left) R -module is injective.

COROLLARY 2.9. *A ring R is semisimple Artinian if and only if every countably generated right R -module is essentially pseudo-injective.*

PROOF. Let M be a cyclic right R -module. Then $(M \oplus R)^{(\mathbb{N})} \cong (M \oplus R)^{(\mathbb{N})} \oplus (M \oplus R)^{(\mathbb{N})}$, which is countably generated, whence essentially pseudo-injective. Thus $(M \oplus R^{(\mathbb{N})})^2$ is essentially pseudo- $(M \oplus R^{(\mathbb{N})})$ -injective. Then by Theorem 2.7, $(M \oplus R^{(\mathbb{N})})$ is quasi-injective, whence R_R -injective. Therefore M is injective. Now the conclusion follows by Osofsky's theorem. \square

COROLLARY 2.10 ([2, Theorem 2.2]). *If $M \oplus N$ is pseudo-injective, then M and N are relatively injective.*

In what follows $E(M)$ stands for the injective hull of M and we will consider M as a submodule of $E(M)$. We will also use the notation $E_N(M)$ for the submodule of $E(M)$ generated by all the isomorphic copies of N . Note that $E_N(M)$ is invariant under monomorphisms of $\text{End}(E(M))$ and that $E_{R_R}(M)$ contains all elements of M with zero right annihilator in R .

PROPOSITION 2.11. *M is essentially pseudo- N -injective if and only if $E_N(M) \subseteq M$.*

PROOF. Assume $E_N(M) \subseteq M$ and let B be an essential submodule of N , and $f : B \rightarrow M$ be a monomorphism. There exists some monomorphism $g : N \rightarrow E(M)$ such that $g|_B = f$. By assumption $g(N) \subseteq M$. Thus g is the desired extension of f , whence M is essentially pseudo- N -injective.

Conversely assume that M is essentially pseudo- N -injective. We will use the same argument as in [10, Lemma 1.13]: Let $h : N \rightarrow E(M)$ be a monomorphism. Let $A = h^{-1}(M)$. Then A is essential in N . Thus, by assumption, the restriction $h|_A$ extends to some $\theta : N \rightarrow M$. Now assume $h(n) \neq \theta(n)$ for some $n \in N$. Then

$x = h(n) - \theta(n) \neq 0$. Since M is essential in $E(M)$, there exists some $r \in R$ such that $0 \neq xr = h(nr) - \theta(nr) \in M$. But then $h(nr) \in M$ so that $nr \in A$. This is a contradiction since $\theta|_A = h|_A$. Now the conclusion follows. \square

COROLLARY 2.12. *M is essentially pseudo-injective if and only if it is invariant under monomorphisms in $\text{End}(E(M))$.*

COROLLARY 2.13. *Let $\{A_i\}$ be a family of submodules of a module N , $B = \Sigma A_i$ and assume M is essentially pseudo- A_i -injective for each i . Then M is essentially pseudo- B -injective.*

PROOF. Let $f : B \rightarrow E(M)$ be a monomorphism. Then $f(B) = \Sigma f(A_i)$. By assumption and Proposition 2.11, $f(B)$ is contained in M . Now the conclusion follows again by Proposition 2.11. \square

The converse of the Corollary 2.13 does not hold in general.

EXAMPLE 3. Let p be a prime. It is easy to see that the \mathbb{Z} -module $\mathbb{Z}/p^2\mathbb{Z}$ is not essentially pseudo- $\mathbb{Z}/p^3\mathbb{Z}$ -injective, but it is trivially essentially pseudo- $(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z})$ -injective.

COROLLARY 2.14. *Let E be an injective module and A be any submodule of E . Then $X = \Sigma\{C \mid C \leq E, C \cong A\}$ is essentially pseudo-injective.*

PROOF. First note that $E(X)$ is a summand of E . As in the proof of Corollary 2.13, for any monomorphism $f : X \rightarrow E(X)$, $f(X)$ is contained in X . The conclusion follows by Proposition 2.11. \square

Goodearl defined a right SI-ring to be one over which every singular right module is injective ([6]). Such rings are precisely right nonsingular rings over which singular right modules are semi-simple (see [3]).

THEOREM 2.15. *Let R be a ring which is either right nonsingular or semi-prime. The following conditions are equivalent:*

- (i) R is a right SI-ring.
- (ii) Any two cyclic singular right R -modules are relatively essentially pseudo-injective.
- (iii) For any two cyclic singular right R -modules B and C , $E_B(C) \subseteq C$.

PROOF. (i) \Rightarrow (ii) Trivial.

(ii) \Leftrightarrow (iii) The statement follows from Proposition 2.11.

(ii) \Rightarrow (i) Assume (ii). Then cyclic singular right R -modules are relatively injective by Proposition 2.4. So if C and M are singular right R -modules and C is cyclic, then C is M -injective by the above argument and [10, Proposition 1.4]. This implies, by [3, Corollary 7.14], that all singular right R -modules are semi-simple.

Now, if R is right nonsingular, the conclusion immediately follows by the preceding remark and the above argument. Else, assume that R is semi-prime. Since singular modules are semi-simple, $Z(R_R)^2 = 0$, whence $Z(R_R) = 0$. Now the conclusion follows by the above argument. \square

3. Pseudo-injectivity

PROPOSITION 3.1 ([16, Corollary 2.9]). *Let M and N be two modules and $X = M \oplus N$. The following conditions are equivalent:*

- (i) M is pseudo- N -injective.
- (ii) For any submodule A of X with $A \cap M = A \cap N = 0$, there exists a submodule T of X containing A with $M \oplus T = X$.

PROOF. (i) \Rightarrow (ii) Assume (i) and let A satisfy the assumptions of (ii). Also let π_M and π_N be as in the Proposition 2.2, and define $\theta : \pi_N(A) \rightarrow \pi_M(A)$ as follows: $\theta(\pi_N(a)) = \pi_M(a)$, for $a \in A$. Then, by assumption, θ extends to some $g \in \text{Hom}(N, M)$. Let $T = \{n + \theta(n) \mid n \in N\}$. Then we have $M \oplus T = X$ and $A \subseteq T$, as required.

(ii) \Rightarrow (i) Assume (ii). Let B be a submodule of N and $f : B \rightarrow M$ be a monomorphism. Call $A = \{b - f(b) \mid b \in B\}$. Then $A \cap M = A \cap N = 0$. Now, by assumption, there exists a submodule T of X containing A with $M \oplus T = X$. Let $\pi : M \oplus T \rightarrow M$ be the obvious projection. Then the restriction $\pi|_N$ is the desired extension of f . \square

Jain and Singh proved in [8, Theorem 3.7] that for a nonsingular module M with finite uniform dimension, the following conditions are equivalent: (i) M is pseudo-injective; (ii) M is invariant under any monomorphism (isomorphism in the terminology of [8]) of $\text{End}(E(M))$ (that is, M is essentially pseudo-injective by Corollary 2.12). The following result extends it to any module with finite uniform dimension.

THEOREM 3.2. *Let M be a module with finite uniform dimension. Assume that for any two essential submodules D and E of M , every isomorphism $h : D \rightarrow E$ can be extended to some $g \in \text{End}(M)$. Then every monomorphism from any submodule of M into M can be extended to a monomorphism of M .*

In particular, a module with finite uniform dimension is pseudo-injective if and only if it is essentially pseudo-injective.

PROOF. Let M be as in the former assumption, A be a submodule of M , and $f : A \rightarrow M$ be a monomorphism. Call $B = f(A)$. Pick, by Zorn's Lemma, two submodules A' and B' of M such that $A \oplus A'$ and $B \oplus B'$ are essential in M . Now, $E(M) = E(A) \oplus E(A') = E(B) \oplus E(B')$ and $E(A) \cong E(B)$. Then by [10, Theorem 1.29] and since M has finite uniform dimension, we have $E(A') \cong E(B')$. Thus A' and B' have isomorphic essential submodules $U \subseteq A'$ and $V \subseteq B'$. Then $A \oplus U$ and $B \oplus V$ are essential submodules of M . And since U and V are isomorphic to each other, there exists an isomorphism $\theta : A \oplus U \rightarrow B \oplus V$ such that $\theta|_A = f$. By assumption θ extends to some monomorphism $g \in \text{End}(M)$. Obviously, $g|_A = f$. Therefore, the conclusion follows. \square

Note that, in [1, Theorem 2.1], Alamelu gives a proof that M is pseudo-injective if and only if M is invariant under monomorphisms of $\text{End}(E(M))$, where M is an arbitrary module over a commutative ring (here the commutativity assumption is irrelevant to the proof). However, the proof is incorrect. In summary, the proof states that for a module M which is invariant under monomorphisms of its injective hull, and for any monomorphism $f : N \rightarrow M$ where N is a submodule of M , f can be extended to a monomorphism $f'' : E(M) \rightarrow E(M)$. This is not correct as the following example shows: Let M be any directly infinite injective module with $M = N \oplus B$, where $M \cong N$ and B is nonzero. Also let $f : N \rightarrow M$ be any isomorphism. Obviously, f cannot be extended to a monomorphism in $\text{End}(E(M))$.

In [4] and [5] Er studied the modules in which isomorphic copies of complements are again complements. These are called SICC-modules in [5]. The following result was proved in [8] for nonsingular modules, but the proof works for an arbitrary pseudo-injective module as well.

LEMMA 3.3 ([8, Lemma 3.1]). *If M is pseudo-injective, then submodules of M isomorphic to complements in M are again complements.*

PROOF. Let K be a complement in M and A be a submodule of M with an isomorphism $f : A \rightarrow K$. Then f extends to some $g \in \text{End}(M)$ by assumption. Pick, by Zorn's Lemma, a complement A' in M essentially containing A . Then the restriction $g|_{A'}$ is obviously a monomorphism. Hence $K = g(A)$ is essential in $g(A')$. Since K is a complement this implies $K = g(A')$, whence $A = A'$. The conclusion follows. \square

REMARK. Modules in which submodules isomorphic to complements are complements always decompose into relatively injective summands by [5, Lemma 4]. So

Corollary 2.10 also follows from that result and Lemma 3.3. It is proved in [2, Corollary 2.8] that a pseudo-injective CS module is continuous. This result also follows from Lemma 3.3 and the definition of CS.

Dinh [2] raised the question whether a CS module M which is pseudo-injective is quasi-injective, and stated in [2] that the answer is affirmative when M is furthermore nonsingular. Now we present some partial answers to Dinh's question.

THEOREM 3.4. *M is quasi-injective if and only if M is pseudo-injective and M^2 is CS.*

PROOF. Assume M is pseudo-injective and M^2 is CS. Let M_1 and M_2 be two isomorphic copies of M and $X = M_1 \oplus M_2$. Note that M is continuous by the preceding remark.

First let A be any complement in X with $A \cap M_1 = 0$ and $A \cap M_2$ essential in A . There exist submodules V and V' of M_2 such that $V \oplus V' = M_2$ and V contains $A \cap M_2$ essentially. Also since M^2 is CS by assumption, we have $A \oplus A' = X$ for some submodule A' of X . Since V is a direct summand of a continuous module, V is continuous (see [10]), whence it has exchange property by [10, Theorem 3.4]. Since $V \cap A$ is essential in A , we have $V \cap A' = 0$. Thus we must have $V \oplus A' = X$. Hence A is isomorphic to a summand, namely V of M_2 .

Now let C be a submodule of X such that $C \cap M_1 = 0$ and pick, by Zorn's Lemma, a complement K in X of M_1 containing C . Again by Zorn's Lemma, choose a complement K_1 in K of $K \cap M_2$ and a complement K_2 in K of K_1 containing $K \cap M_2$. Note that $K \cap M_2$ is essential in K_2 and that K_1 and K_2 are complements in X by [3, 1.10]. By Proposition 3.1 there exists some submodule T of X containing K_1 with $M_1 \oplus T = X$. Then $T \cong M$ and K_1 is a complement in T , whence K_1 is isomorphic to a complement in M_2 . Also by the preceding paragraph K_2 is isomorphic to a complement of M_2 too. Now consider the usual projection $\pi : M_1 \oplus M_2 \rightarrow M_2$. We have $M_1 \oplus (K_1 \oplus K_2) = M_1 \oplus (\pi(K_1) \oplus \pi(K_2))$, where $\pi(K_i) \cong K_i$. Hence by continuity of M_2 and the above argument, $\pi(K_1) \oplus \pi(K_2)$ is a summand of M_2 . Now, since K is a complement of M_1 , $M_1 \oplus K = M_1 \oplus \pi(K)$ is essential in X . Then $\pi(K)$ is essential in M_2 . Also, by choice of K_i , $K_1 \oplus K_2$ is essential in K . Then $\pi(K_1) \oplus \pi(K_2)$ is essential in $\pi(K)$, hence in M_2 . This implies that $M_2 = \pi(K_1) \oplus \pi(K_2) = \pi(K)$. Thus $M_1 \oplus K = X$. Now it follows by [3, Lemma 7.5] that M_1 is M_2 -injective. The proof is now complete. \square

The following is a key result.

LEMMA 3.5. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of uniform modules M_i . M is quasi-injective if and only if it is pseudo-injective. In particular, any uniform pseudo-injective module is quasi-injective.*

PROOF. First let M be a uniform pseudo-injective module. Let A be a submodule of M and $f : A \rightarrow M$ be a nonzero homomorphism. If $\text{Ker}(f) = 0$, then f can be extended to an element of $\text{End}(M)$ by assumption. So assume $\text{Ker}(f) \neq 0$. Let $\delta = i_A - f$, where $i_A : A \rightarrow M$ is the inclusion map. Since $\text{Ker}(f) \neq 0$ and M is uniform, $\text{Ker}(\delta) = 0$. Then by pseudo-injectivity assumption δ can be extended to some $g \in \text{End}(M)$. Now $1 - g$ is obviously an extension of f . Thus M is quasi-injective.

Now let $M = \bigoplus_{i \in I} M_i$ be a direct sum of uniform modules M_i and assume that M is pseudo-injective. Then, by Corollary 2.10, $M(I - i)$ is M_i -injective for all $i \in I$. Now by the preceding paragraph and since direct summands of pseudo-injectives are obviously pseudo-injective, each M_i is quasi-injective. Therefore M is quasi-injective. \square

THEOREM 3.6. *Over a right Noetherian ring R , a right R -module M is quasi-injective if and only if M is a pseudo-injective CS-module.*

PROOF. Let M be a pseudo-injective CS module. Then M is a direct sum of uniform submodules by [11]. Now the result follows by Lemma 3.5. \square

Before proving the next result, note that R is called a right countably Σ -CS ring if $R_R^{(\mathbb{N})}$ is a CS module.

THEOREM 3.7. *The following conditions are equivalent for a ring R :*

- (i) R is a quasi-Frobenius ring.
- (ii) Every projective right R -module is essentially pseudo- R_R -injective.
- (iii) $R_R^{(\mathbb{N})}$ is essentially pseudo- R_R -injective.
- (iv) R is a right countably Σ -CS ring with finite uniform dimension and R_R is essentially pseudo-injective.

PROOF. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are obvious, and (i) \Rightarrow (iv) follows from the fact that every injective module is CS, and (iii) \Rightarrow (i) follows by Theorem 2.7.

(iv) \Rightarrow (i) Since R_R has finite uniform dimension, then R_R is pseudo-injective by Theorem 3.2. Then by Theorem 3.4 R is a right self-injective ring with finite uniform dimension. Hence R is a semiperfect right countably Σ -CS ring. This implies by [7] that R is Artinian. Now the conclusion follows. \square

The following results were proved in [5, Theorem 2, Corollary 4, Theorem 3, Theorem 4] for modules in which submodules isomorphic to complements are complements. Each pseudo-injective module satisfies this property by Lemma 3.3, whence we have the following corollaries.

COROLLARY 3.8. *Any decomposition of a pseudo-injective module into indecomposable submodules complements summands.*

COROLLARY 3.9. *An essentially pseudo-injective module with finite uniform dimension has the internal cancellation property.*

Recall that every right R -module over a right Noetherian ring R is locally Noetherian.

COROLLARY 3.10. *If M is a locally Noetherian pseudo-injective module, then $M = A \oplus B$, where A is a maximal quasi-injective summand, B has no quasi-injective summands, and A and B have no nonzero isomorphic submodules.*

COROLLARY 3.11. *A locally Noetherian Dedekind-finite pseudo-injective module has internal cancellation property.*

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