ON $\psi$-DIRECT SUMS OF BANACH SPACES AND CONVEXITY

MIKIO KATO, KICHI-SUKE SAITO and TAKAYUKI TAMURA

Dedicated to Maestro Ivry Gitlis on his 80th birthday with deep respect and affection

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Abstract

Let $X_1, X_2, \ldots, X_N$ be Banach spaces and $\psi$ a continuous convex function with some appropriate conditions on a certain convex set in $\mathbb{R}^{N-1}$. Let $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ be the direct sum of $X_1, X_2, \ldots, X_N$ equipped with the norm associated with $\psi$. We characterize the strict, uniform, and locally uniform convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ by means of the convex function $\psi$. As an application these convexities are characterized for the $\ell_p$-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{\psi, q}$ ($1 < q \leq p \leq \infty$, $q < \infty$), which includes the well-known facts for the $\ell_p$-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_p$ in the case $p = q$.


Keywords and phrases: absolute norm, convex function, direct sum of Banach spaces, strictly convex space, uniformly convex space, locally uniformly convex space.

1. Introduction and preliminaries

A norm $\| \cdot \|$ on $\mathbb{C}^N$ is called absolute if $\|(z_1, \ldots, z_N)\| = \|(|z_1|, \ldots, |z_N|)\|$ for all $(z_1, \ldots, z_N) \in \mathbb{C}^N$, and normalized if $\|(1, 0, \ldots, 0)\| = \cdots = \|(0, \ldots, 0, 1)\| = 1$ (see for example [3, 2]). In case of $N = 2$, according to Bonsall and Duncan [3] (see also [12]), for every absolute normalized norm $\| \cdot \|$ on $C^2$ there corresponds a unique continuous convex function $\psi$ on the unit interval $[0, 1]$ satisfying

$$\max\{1 - t, t\} \leq \psi(t) \leq 1$$

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under the equation $\psi(t) = \|(1 - t, t)\|$. Recently in [11] Saito, Kato and Takahashi presented the $N$-dimensional version of this fact, which states that for every absolute normalized norm $\| \cdot \|$ on $\mathbb{C}^N$ there corresponds a unique continuous convex function $\psi$ satisfying some appropriate conditions on the convex set

$$\Delta_N = \left\{ t = (t_1, \ldots, t_{N-1}) \in \mathbb{R}^{N-1} : \sum_{j=1}^{N-1} t_j \leq 1, t_j \geq 0 \right\}$$

under the equation $\psi(t) = \left\| (1 - \sum_{j=1}^{N-1} t_j, t_1, \ldots, t_{N-1}) \right\|$.

For an arbitrary finite number of Banach spaces $X_1, X_2, \ldots, X_N$, we define the $\psi$-direct sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{\psi}$ to be their direct sum equipped with the norm

$$\|(x_1, x_2, \ldots, x_N)\|_{\psi} = \| (\|x_1\|, \|x_2\|, \ldots, \|x_N\|)\|_{\psi} \quad \text{for} \quad x_j \in X_j,$$

where $\| \cdot \|_{\psi}$ term in the right-hand side is the absolute normalized norm on $\mathbb{C}^N$ with the corresponding convex function $\psi$. This extends the notion of $\ell_p$-sum of Banach spaces. The aim of this paper is to characterize the strict, and uniform convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{\psi}$. The locally uniform convexity is also included. For the case $N = 2$, the first two have been recently proved in Takahashi-Kato-Saito [13] and Saito-Kato [10], respectively. However the proof of the uniform convexity for the 2-dimensional case given in [10] seems difficult to be extended to the $N$-dimensional case, though it is of independent interest as it is of real analytic nature and maybe useful for estimating the modulus of convexity. Our proof for the $N$-dimensional case is essentially different from that in [10]. As an application we shall consider the $\ell_{p,q}$-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ and show that $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ is uniformly convex if and only if, whenever $1 < q \leq p \leq \infty$, $q < \infty$. The same is true for the strict and locally uniform convexity. These results include the well-known facts for the $\ell_p$-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_p$ as the case $p = q$.

Let us recall some definitions. A Banach space $X$ or its norm $\| \cdot \|$ is called strictly convex if $\|x\| = \|y\| = 1$ ($x \neq y$) implies $\|(x + y)/2\| < 1$. This is equivalent to the following statement: if $\|x + y\| = \|x\| + \|y\|$, $x \neq 0$, $y \neq 0$, then $x = \lambda y$ with some $\lambda > 0$ (see for example [9, page 432], [1]). $X$ is called uniformly convex provided for any $\epsilon$ ($0 < \epsilon < 2$) there exists $\delta > 0$ such that whenever $\|x - y\| \geq \epsilon$. $\|x\| = \|y\| = 1$, one has $\|(x + y)/2\| \leq 1 - \delta$, or equivalently, provided for any $\epsilon$ ($0 < \epsilon < 2$) one has $\delta_X(\epsilon) > 0$, where $\delta_X$ is the modulus of convexity of $X$, that is,

$$\delta_X(\epsilon) := \inf\{1 - \|(x + y)/2\| ; \|x - y\| \geq \epsilon, \|x\| = \|y\| = 1\} \quad (0 \leq \epsilon \leq 2).$$

We also have the following restatement: $X$ is uniformly convex if and only if, whenever $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \to 1$, it follows that $\|x_n - y_n\| \to 0$. $X$ is called locally uniformly convex (see for example [9, 4]) if for any $x \in X$ with $\|x\| = 1$ and
for any $\epsilon$ such that $0 < \epsilon < 2$ there exists $\delta > 0$ such that if $\|x - y\| \geq \epsilon$, $\|y\| = 1$, then $\|(x + y)/2\| \leq 1 - \delta$. Clearly the notion of locally uniform convexity is between those of uniform and strict convexities.

2. Absolute norms on $\mathbb{C}^N$ and $\psi$-direct sums $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{\Psi}$

Let $AN_N$ denote the family of all absolute normalized norms on $\mathbb{C}^N$. Let

$$\Delta_N = \{(s_1, s_2, \ldots, s_{N-1}) \in \mathbb{R}^{N-1} : s_1 + s_2 + \cdots + s_{N-1} \leq 1, s_j \geq 0 (\forall j)\}.$$

For any $\| \cdot \| \in AN_N$ define the function $\psi$ on $\Delta_N$ by

(1) $\psi(s) = \|(1 - s_1 - \cdots - s_{N-1}, s_1, \ldots, s_{N-1})\|$ for $s = (s_1, \ldots, s_{N-1}) \in \Delta_N$.

Then $\psi$ is continuous and convex on $\Delta_N$, and satisfies the following conditions:

(A$_0$) $\psi(0, \ldots, 0) = \psi(1, 0, \ldots, 0) = \cdots = \psi(0, \ldots, 0, 1) = 1$,

(A$_1$) $\psi(s_1, \ldots, s_{N-1}) \geq (s_1 + \cdots + s_{N-1})\psi\left(\frac{s_1}{\sum_{i=1}^{N-1} s_i}, \ldots, \frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i}\right),$

(A$_2$) $\psi(s_1, \ldots, s_{N-1}) \geq (1 - s_1)\psi\left(0, \frac{s_2}{1 - s_1}, \ldots, \frac{s_{N-1}}{1 - s_1}\right),$ 

(A$_N$) $\psi(s_1, \ldots, s_{N-1}) \geq (1 - s_{N-1})\psi\left(\frac{s_1}{1 - s_{N-1}}, \ldots, \frac{s_{N-2}}{1 - s_{N-1}}, 0\right).$

Note that from (A$_0$) it follows that $\psi(s_1, \ldots, s_{N-1}) \leq 1$ on $\Delta_N$ as $\psi$ is convex. Denote $\Psi_N$ be the family of all continuous convex functions $\psi$ on $\Delta_N$ satisfying (A$_0$), (A$_1$), (A$_2$), (A$_N$). Then the converse holds true: For any $\psi \in \Psi_N$ define

(2) $\|z_1, \ldots, z_N\|_{\psi} = \left\{\begin{array}{ll}
\left(\sum_{i=1}^{N} |z_i|\right) \psi\left(|z_1|/\left(\sum_{i=1}^{N} |z_i|\right), \ldots, |z_N|/\left(\sum_{i=1}^{N} |z_i|\right)\right) & \text{if } (z_1, \ldots, z_N) \neq (0, \ldots, 0), \\
0 & \text{if } (z_1, \ldots, z_N) = (0, \ldots, 0).
\end{array}\right.$

Then $\| \cdot \|_\psi \in AN_N$ and $\| \cdot \|_\psi$ satisfies (1). Thus the families $AN_N$ and $\Psi_N$ are in one-to-one correspondence under equation (1) (Saito-Kato-Takahashi [11, Theorem 4.2]). The $\ell_p$-norms

$$\|(z_1, \ldots, z_N)\|_p = \left\{\begin{array}{ll}
|z_1|^p + \cdots + |z_N|^p & \text{if } 1 \leq p < \infty, \\
\max\{|z_1|, \ldots, |z_N|\} & \text{if } p = \infty.
\end{array}\right.$$
are typical examples of absolute normalized norms, and for any \( \| \cdot \| \in \mathcal{AN} \), we have

\[
(3) \quad \| \cdot \|_\infty \leq \| \cdot \| \leq \| \cdot \|_1
\]

([[11], Lemma 3.1], see also [3]). The functions corresponding to \( \ell_p \)-norms on \( \mathbb{C}^N \) are

\[
\psi_p(s_1, \ldots, s_{N-1}) = \left\{ \begin{array}{ll}
\left( \left(1 - \sum_{j=1}^{N-1} s_j \right)^p + \sum_{j=1}^{N-1} s_j^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\
\max \left\{ 1 - \sum_{j=1}^{N-1} s_j, s_1, \ldots, s_{N-1} \right\} & \text{if } p = \infty
\end{array} \right.
\]

for \((s_1, \ldots, s_{N-1}) \in \Delta_N\).

Let \( X_1, X_2, \ldots, X_N \) be Banach spaces. Let \( \psi \in \Psi_N \) and let \( \| \cdot \|_\psi \) be the corresponding norm in \( \mathcal{AN} \). Let \( (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi \) be the direct sum of \( X_1, X_2, \ldots, X_N \) equipped with the norm

\[
(4) \quad \| (x_1, x_2, \ldots, x_N) \|_\psi := \| (\| x_1 \|, \| x_2 \|, \ldots, \| x_N \|) \|_\psi \quad \text{for } x_j \in X_j.
\]

As is immediately seen, \( (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi \) is a Banach space.

**Example.** Let \( 1 \leq q \leq p \leq \infty \), \( q < \infty \). We consider the Lorentz \( \ell_{p,q} \)-norm

\[
\| z \|_{p,q} = \{ \sum_{j=1}^{N} j^{q/(p-1)} z_j^q \}^{1/q} \quad \text{for } z = (z_1, \ldots, z_N) \in \mathbb{C}^N,
\]

where \( \{z_j^*\} \) is the non-increasing rearrangement of \( \{|z_j|\} \), that is, \( z_1^* \geq z_2^* \geq \cdots \geq z_N^* \). (Note that in case of \( 1 \leq p < q \leq \infty \), \( \| \cdot \|_{p,q} \) is not a norm but a quasi-norm (see [6, Proposition 1], [14, page 126])). Evidently \( \| \cdot \|_{p,q} \in \mathcal{AN} \) and the corresponding convex function \( \psi_{p,q} \) is obtained by

\[
(5) \quad \psi_{p,q}(s) = \| (1 - s_1 - \cdots - s_{N-1}, s_1, \ldots, s_{N-1}) \|_{p,q}
\]

for \((s_1, \ldots, s_{N-1}) \in \Delta_N\), that is, \( \| \cdot \|_{p,q} = \| \cdot \|_{\psi_{p,q}} \). Let \( (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q} \) be the direct sum of Banach spaces \( X_1, X_2, \ldots, X_N \) equipped with the norm

\[
\| (x_1, \ldots, x_N) \|_{p,q} := \| (\| x_1 \|, \ldots, \| x_N \|) \|_{p,q},
\]

we call it the \( \ell_{p,q} \)-sum of \( X_1, X_2, \ldots, X_N \). If \( p = q \) the \( \ell_{p,p} \)-sum is the usual \( \ell_p \)-sum \( (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_p \).

For some other examples of absolute norms on \( \mathbb{C}^N \) we refer the reader to [11] (see also [12]).

### 3. Strict convexity of \( (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi \)

A function \( \psi \) on \( \Delta_N \) is called strictly convex if for any \( s, t \in \Delta_N (s \neq t) \) one has \( \psi((s + t)/2) < (\psi(s) + \psi(t))/2 \). For absolute norms on \( \mathbb{C}^N \), we have
Lemma 3.1 (Saito-Kato-Takahashi [11, Theorem 4.2]). Let \( \psi \in \Psi_N \). Then \((\mathbb{C}^N, \| \cdot \|_\psi)\) is strictly convex if and only if \( \psi \) is strictly convex.

The following lemma concerning the monotonicity property of the absolute norms on \( \mathbb{C}^N \) is useful in the sequel.

Lemma 3.2 (Saito-Kato-Takahashi [11, Lemma 4.1]). Let \( \psi \in \Psi_N \). Let \( z = (z_1, \ldots, z_N) \), \( w = (w_1, \ldots, w_N) \in \mathbb{C}^N \).

(i) If \( |z_j| \leq |w_j| \) for all \( j \), then \( \| z \|_\psi \leq \| w \|_\psi \).

(ii) Let \( \psi \) be strictly convex. If \( |z_j| \leq |w_j| \) for all \( j \) and \( |z_j| < |w_j| \) for some \( j \), then \( \| z \|_\psi < \| w \|_\psi \).

Theorem 3.3. Let \( X_1, X_2, \ldots, X_N \) be Banach spaces and let \( \psi \in \Psi_N \). Then \((X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi\) is strictly convex if and only if \( X_1, X_2, \ldots, X_N \) are strictly convex and \( \psi \) is strictly convex.

Proof. Let \((X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi\) be strictly convex. Then, each \( X_j \) and \((\mathbb{C}^N, \| \cdot \|_\psi)\) are strictly convex since they are isometrically imbedded into \((X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi\). According to Lemma 3.1, \( \psi \) is strictly convex.

Conversely, let each \( X_j \) and \( \psi \) be strictly convex. Take arbitrary \( x = (x_j) \), \( y = (y_j) \), \( x \neq y \) in \((X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi\) with \( \| x \|_\psi = \| y \|_\psi = 1 \). Let first \((\| x_1 \|, \ldots, \| x_N \|) = (\| y_1 \|, \ldots, \| y_N \|)\). Then, if \( \| x + y \|_\psi = 2 \),

\[
2 = \| x + y \|_\psi = \| (x_1 + y_1, \ldots, x_N + y_N) \|_\psi \\
\leq \| (x_1 + y_1, \ldots, x_N + y_N) \|_\psi \\
\leq \| x \|_\psi + \| y \|_\psi = 2,
\]

from which it follows that \( \| x_j + y_j \| = \| x_j \| + \| y_j \| \) for all \( j \) by Lemma 3.2. As each \( X_j \) is strictly convex, \( x_j = k_j y_j \) with \( k_j > 0 \). Since \( \| x_j \| = \| y_j \| \), we have \( k_j = 1 \) and hence \( x_j = y_j \) for all \( j \), or \( x = y \), which is a contradiction. Therefore we have \( \| x + y \|_\psi < 2 \). Let next \((\| x_1 \|, \ldots, \| x_N \|) \neq (\| y_1 \|, \ldots, \| y_N \|)\). Since \( \psi \) is strictly convex, \((\mathbb{C}^N, \| \cdot \|_\psi)\) is strictly convex by Lemma 3.1. Consequently we have

\[
\| x + y \|_\psi = \| (x_1 + y_1, \ldots, x_N + y_N) \|_\psi \\
\leq \| (x_1 + y_1, \ldots, x_N + y_N) \|_\psi \\
= \| (x_1, \ldots, x_N) + (y_1, \ldots, y_N) \|_\psi < 2,
\]

as is desired. \( \square \)

Now we see that the function \( \psi_{p,q} \) in the above example is strictly convex if \( 1 < q \leq p \leq \infty, q < \infty \). We need the next lemma.
**Lemma 3.4 ([5]).** Let \( \{\alpha_j\}, \{\beta_j\} \in \mathbb{R}^N \) and \( \alpha_j \geq 0, \beta_j \geq 0 \). Let \( \{\alpha_j^*\}, \{\beta_j^*\} \) be their non-increasing rearrangements, that is, \( \alpha_j^* \geq \alpha_j^* \geq \cdots \geq \alpha_N^* \) and \( \beta_j^* \geq \beta_j^* \geq \cdots \geq \beta_N^* \). Then \( \sum_{j=1}^N \alpha_j \beta_j \leq \sum_{j=1}^N \alpha_j^* \beta_j^* \).

**Proposition 3.5.** Let \( 1 < q \leq p \leq \infty, \ q < \infty \). Then the function \( \psi_{p,q} \) given by (5) is strictly convex on \( N \).

**Proof.** Let \( s = (s_j), t = (t_j) \in N, s \neq t \). Without loss of generality we may assume that

\[
2 = (s_1 + t_1) - \cdots - (s_{N-1} + t_{N-1}) \geq s_1 + t_1 \geq \cdots \geq s_{N-1} + t_{N-1} \geq 0.
\]

Put

\[
\sigma = (1 - s_1 - \cdots - s_{N-1}, 2^{1/p-1/q} s_1, \ldots, N^{1/p-1/q} s_{N-1}),
\]

\[
\tau = (1 - t_1 - \cdots - t_{N-1}, 2^{1/p-1/q} t_1, \ldots, N^{1/p-1/q} t_{N-1}).
\]

Then by Lemma 3.4 we have

\[
\|\sigma\|_q = \left( (1 - s_1 - \cdots - s_{N-1})^q + 2^{q/p-1} s_1^q + \cdots + N^{q/p-1} s_{N-1}^q \right)^{1/q} \leq \|(1 - s_1 - \cdots - s_{N-1}, s_1, \ldots, s_{N-1})\|_{p,q} = \psi_{p,q}(s)
\]

and \( \|\tau\|_q \leq \psi_{p,q}(t) \). On the other hand,

\[
\psi_{p,q}\left( \frac{s + t}{2} \right) = \left\{ \left( 1 - \sum_{i=1}^{N-1} \frac{s_i + t_i}{2} \right) + \sum_{i=1}^{N-1} (i + 1)^{q/p-1} \left( \frac{s_i + t_i}{2} \right) \right\}^{1/q} \leq \left\{ \left( 1 - \sum_{i=1}^{N-1} s_i \right) + \left( 1 - \sum_{i=1}^{N-1} t_i \right) \right\}^{1/q} = \frac{\|\sigma + \tau\|_q}{2}.
\]

Since \( \ell_q \)-norm \( \|\cdot\|_q (1 < q < \infty) \) is strictly convex and \( s \neq t \), we have \( \|\sigma + \tau\|_q < \|\sigma\|_q + \|\tau\|_q \). Indeed, if \( \|\sigma + \tau\|_q = \|\sigma\|_q + \|\tau\|_q \), then \( \sigma = k \tau \) with some \( k > 0 \) (note that \( \sigma \neq 0, \tau \neq 0 \)). Hence \( s_j = kt_j \) for all \( j \), and \( 1 - \sum_{i=1}^{N-1} s_i = k \left( 1 - \sum_{i=1}^{N-1} t_i \right) \).

Therefore, \( k = 1 \) and we have \( s = t \), which is a contradiction. Consequently,

\[
\psi_{p,q}\left( \frac{s + t}{2} \right) = \frac{\|\sigma + \tau\|_q}{2} < \frac{\|\sigma\|_q + \|\tau\|_q}{2} \leq \psi_{p,q}(s) + \psi_{p,q}(t),
\]

or \( \psi_{p,q} \) is strictly convex. \( \square \)

By Theorem 3.3 and Proposition 3.5 we have the following result for the \( \ell_{p,q} \)-sum of Banach spaces.
Corollary 3.6. Let \( 1 < q < p \leq \infty, \ q < \infty \). Then, \( \ell_{q,p}-\text{sum} (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q} \) is strictly convex if and only if \( X_1, X_2, \ldots, X_N \) are strictly convex.

In particular, the \( \ell_{p,1}-\text{sum} (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p} \), \( 1 < p < \infty \), is strictly convex if and only if \( X_1, X_2, \ldots, X_N \) are strictly convex.

4. Uniform convexity of \( (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi \)

Let us characterize the uniform convexity of \( (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi \).

Theorem 4.1. Let \( X_1, X_2, \ldots, X_N \) be Banach spaces and let \( \psi \in \Psi_N \). Then \( (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi \) is uniformly convex if and only if \( X_1, X_2, \ldots, X_N \) are uniformly convex and \( \psi \) is strictly convex.

Proof: The necessity assertion is proved in the same way as the proof of Theorem 3.3. Assume that \( X_1, X_2, \ldots, X_N \) are uniformly convex and \( \psi \) is strictly convex.

Take an arbitrary \( \epsilon > 0 \) and put

\[
\delta := 2\delta_X(\epsilon) = \inf \{ 2 - \| x + y \|_\psi : \| x - y \|_\psi \geq \epsilon, \| x \|_\psi = \| y \|_\psi = 1 \}.
\]

We show that \( \delta > 0 \). There exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi \) so that

\[
\| x_n - y_n \|_\psi \geq \epsilon, \quad \| x_n \|_\psi = \| y_n \|_\psi = 1
\]

and

\[
\lim_{n \to \infty} \| x_n + y_n \|_\psi = 2 - \delta.
\]

Let \( x_n = (x_n^{(1)}, \ldots, x_n^{(N)}) \) and \( y_n = (y_n^{(1)}, \ldots, y_n^{(N)}) \). Since for each \( 1 \leq j \leq N \), \( \| x_j^{(n)} \| = \| 0, \ldots, 0, x_j^{(n)}, 0, \ldots, 0 \|_\psi \leq \| x_n \|_\psi = 1 \) and \( \| y_j^{(n)} \| = \| y_n \|_\psi = 1 \) for all \( n \), the sequences \( \{ \| x_j^{(n)} \| \} \) and \( \{ \| y_j^{(n)} \| \} \), have a convergent subsequence respectively. So we may assume that \( \| x_j^{(n)} \| \to a_j, \| y_j^{(n)} \| \to b_j \) as \( n \to \infty \). Further, in the same way, we may assume that

\[
\| x_j^{(n)} - y_j^{(n)} \| \to c_j \quad \text{as} \quad n \to \infty
\]

and

\[
\| x_j^{(n)} + y_j^{(n)} \| \to d_j \quad \text{as} \quad n \to \infty.
\]

Put \( K_n = \sum_{j=1}^{N} \| x_j^{(n)} \| \). Then \( \| x_n \|_\psi = K_n \psi (\| x_1^{(n)} \| / K_n, \ldots, \| x_N^{(n)} \| / K_n) = 1 \). Letting \( n \to \infty \), as \( \psi \) is continuous, we have

\[
\| (a_1, \ldots, a_N) \|_\psi = \left( \sum_{j=1}^{N} a_j \right) \psi \left( \frac{a_2}{\sum_{j=1}^{N} a_j}, \ldots, \frac{a_N}{\sum_{j=1}^{N} a_j} \right) = 1.
\]
Also we have

\[
\| (b_1, \ldots, b_N) \|_\psi = \left( \sum_{j=1}^N b_j \right) \psi \left( \frac{b_2}{\sum_{j=1}^N b_j}, \ldots, \frac{b_N}{\sum_{j=1}^N b_j} \right) = 1.
\]

Next let \( n \to \infty \) in (6), or in

\[
\| x_n - y_n \|_\psi = \left( \sum_{j=1}^N \| x_j^{(n)} - y_j^{(n)} \| \right) \times \psi \left( \frac{\| x_2^{(n)} - y_2^{(n)} \|}{\sum_{j=1}^N \| x_j^{(n)} - y_j^{(n)} \|}, \ldots, \frac{\| x_N^{(n)} - y_N^{(n)} \|}{\sum_{j=1}^N \| x_j^{(n)} - y_j^{(n)} \|} \right) \geq \epsilon.
\]

Then we have

\[
\| (c_1, \ldots, c_N) \|_\psi = \left( \sum_{j=1}^N c_j \right) \psi \left( \frac{c_2}{\sum_{j=1}^N c_j}, \ldots, \frac{c_N}{\sum_{j=1}^N c_j} \right) \geq \epsilon
\]

by (8). In the same way, according to (7) and (9), we have

\[
\| (d_1, \ldots, d_N) \|_\psi = 2 - \delta.
\]

Now, assume that \( (a_1, \ldots, a_N) \neq (b_1, \ldots, b_N) \). Then, according to (10), (11) and the strict convexity of \( \psi \) we obtain that

\[
2 - \delta = \| (d_1, \ldots, d_N) \|_\psi \leq \| (a_1 + b_1, \ldots, a_N + b_N) \|_\psi < 2,
\]

which implies \( \delta > 0 \). Next, let \( (a_1, \ldots, a_N) = (b_1, \ldots, b_N) \). Since \( (c_1, \ldots, c_N) \neq (0, \ldots, 0) \) from (12), we may assume that \( c_1 > 0 \) without loss of generality. Then as

\[
c_1 = \lim_{n \to \infty} \| x_1^{(n)} - y_1^{(n)} \| \leq \lim_{n \to \infty} \left( \| x_1^{(n)} \| + \| y_1^{(n)} \| \right) = a_1 + b_1 = 2a_1,
\]

we have \( a_1 > 0 \) and

\[
0 < \frac{c_1}{a_1} = \lim_{n \to \infty} \frac{\| x_1^{(n)} \|}{\| x_1^{(n)} - y_1^{(n)} \|} = \lim_{n \to \infty} \frac{\| x_1^{(n)} \|}{\| y_1^{(n)} \|} = 1.
\]

Indeed, we have the latter identity because

\[
\left\| \frac{x_1^{(n)}}{\| x_1^{(n)} \|} - \frac{y_1^{(n)}}{\| y_1^{(n)} \|} \right\| \leq \| y_1^{(n)} \| \left( \frac{1}{\| x_1^{(n)} \|} - \frac{1}{\| y_1^{(n)} \|} \right) \rightarrow b_1 \left| \frac{1}{a_1} - \frac{1}{b_1} \right| = 0 \quad \text{as} \ n \to \infty.
\]
Since $X_1$ is uniform convex, it follows from (14) that

$$
\frac{d_1}{d_1} = \lim_{n \to \infty} \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} + \frac{y_1^{(n)}}{\|y_1^{(n)}\|} \right\| = \lim_{n \to \infty} \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} + \frac{y_1^{(n)}}{\|y_1^{(n)}\|} \right\| < 2,
$$

whence $d_1 < 2a_1$. Accordingly, by (13) and Lemma 3.2 we obtain that

$$
2 - \delta = \|(d_1, d_2, \ldots, d_N)\|_\psi < \|(2a_1, a_2 + b_2, \ldots, a_N + b_N)\|_\psi = \|(a_1 + b_1, a_2 + b_2, \ldots, a_N + b_N)\|_\psi \leq \|(a_1, \ldots, a_N)\|_\psi + \|(b_1, \ldots, b_N)\|_\psi = 2,
$$

which implies $\delta > 0$. This completes the proof.

The parallel argument works for the locally uniform convexity and we obtain the next result.

**Theorem 4.2.** Let $\psi \in \Psi_N$. Then $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ is locally uniformly convex if and only if $X_1, X_2, \ldots, X_N$ are locally uniformly convex and $\psi$ is strictly convex.

Indeed, for the sufficiency, take an arbitrary $x \in (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ with $\|x\|_\psi = 1$ and merely let $x_n = x$ in the above proof. By Theorem 4.1 and Theorem 4.2 combined with Proposition 3.5 we obtain the following corollary.

**Corollary 4.3.** Let $1 < q \leq p \leq \infty$, $q < \infty$. Then, $\ell_{p,q}$-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ is uniformly convex (locally uniformly convex) if and only if $X_1, X_2, \ldots, X_N$ are uniformly convex (locally uniformly convex).

In particular, the $\ell_p$-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p}$, $1 < p < \infty$, is uniformly convex (locally uniformly convex) if and only if $X_1, X_2, \ldots, X_N$ are uniformly convex (locally uniformly convex).

**References**


Department of Mathematics
Kyushu Institute of Technology
Kitakyushu 804-8550
Japan
e-mail: katom@tobata.isc.kyutech.ac.jp

Department of Mathematics
Faculty of Science
Niigata University
Niigata 950-2181
Japan
e-mail: saito@math.sc.niigata-u.ac.jp

Graduate School of Social Sciences and Humanities
Chiba University
Chiba 263-8522
Japan
e-mail: tamura@le.chiba-u.ac.jp