MULTIPlicITIES OF HIGHER LIE CHARACTERS

MANFRED SCHOCKER

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Abstract

The higher Lie characters of the symmetric group $S_n$ arise from the Poincaré-Birkhoff-Witt basis of the free associative algebra. They are indexed by the partitions of $n$ and sum up to the regular character of $S_n$. A combinatorial description of the multiplicities of their irreducible components is given. As a special case the Kraskiewicz-Weyman result on the multiplicities of the classical Lie character is obtained.


Keywords and phrases: symmetric group, general linear group, free Lie algebra, tableau, major index.

1. Introduction

At the beginning of the last century Schur studied the structure of the tensor algebra $T(V)$ over a finite dimensional $K$-vector space $V$ as a $GL(V)$-module. In his thesis ([13]) and a famous subsequent paper ([14]) he was able to describe the decomposition of the homogeneous components

$$T_n(V) := V \otimes \cdots \otimes V,$$

of degree $n$ in $T(V)$ into irreducible $GL(V)$-modules using the irreducible representations of the symmetric group $S_n$. The usual Lie bracketing $[x, y] := xy - yx$ turns $T(V)$ into a Lie algebra. The Lie subalgebra $L(V)$ generated by $V$ is free over any basis of $V$ by a classical result of Witt ([17]), and $L_n(V) := T_n(V) \cap L(V)$ is a $GL(V)$-submodule of $T_n(V)$ for all $n$. Let $q = q_1 \ldots q_k$ be a partition of $n$, that is, $q_1 \geq \cdots \geq q_k$ and $q_1 + \cdots + q_k = n$. Then we define

$$L_q(V) := \left\{ \sum_{\pi \in \mathcal{S}_n} P_{1, \pi} \cdots P_{k, \pi} \mid P_i \in L_{q_i}(V) \text{ for } 1 \leq i \leq k \right\},$$

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By the Poincaré-Birkhoff-Witt theorem, \( T_n(V) \) is the direct sum of these subspaces:

\[
T_n(V) = \bigoplus_{q \vdash n} L_q(V),
\]

and this decomposition is \( GL(V) \)-invariant.

Meanwhile, different families of idempotents \( e_q \) in the group algebra \( KS_n \) indexed by partitions have been introduced such that \( L_q(V) \equiv e_q T_n(V) \) for all \( q \) (see, for example, [2, 3, 11]). For any decomposition \( e_q KS_n = \bigoplus_p a_{q,p} M_p \) into irreducible \( S_n \)-modules, we now have

\[
L_q(V) = e_q T_n(V) \equiv e_q KS_n \otimes_{KS_n} T_n(V) = \bigoplus_p a_{q,p} (M_p \otimes_{KS_n} T_n(V)).
\]

In this decomposition, by Schur’s fundamental result, \( M_p \otimes_{KS_n} T_n(V) \) is either 0 or an irreducible \( GL(V) \)-module. Hence the \( GL(V) \)-module structure of \( L_q(V) \) is completely determined by the multiplicities \( a_{q,p} \) of the higher Lie module \( e_q KS_n \) of \( S_n \). In this vein, for the special case of \( q = n \), the problem of describing the \( GL(V) \)-module structure of \( L_n(V) \) formulated by Thrall ([16]) could finally be solved in a satisfying way by works of Klyachko ([8]) and Krasikiewicz and Weyman ([9]).

The higher Lie characters \( \lambda_q \) of \( S_n \) corresponding to the modules \( e_q KS_n \) sum up to the regular character of \( S_n \), by (1), and it is natural to ask for their multiplicities for arbitrary \( q \). In this paper, a combinatorial description of these multiplicities is given in terms of alternating sums of numbers of standard tableaux with certain major index properties (Section 3). For \( q = n \), we obtain the Krasikiewicz-Weyman result mentioned above. Our approach is based on a generalization of Klyachko’s result (Section 2) combined with the calculus of noncommutative character theory introduced in [6] (Section 4).

2. The reduction to partitions of block type

Let \( q \) be a partition of \( n \). The higher Lie character \( \lambda_q \) is induced by a certain linear character of the centralizer of an element of cycle type \( q \) in \( S_n \). For \( q = n \), this result is due to Klyachko ([8]). In full generality, it is implicitly contained in [1] for the first time (for details, see [12, Section 8.5]) and will be briefly recalled in two steps in this section.

Let \( \mathbb{N} \) (\( \mathbb{N}_0 \), respectively) be the set of all positive (nonnegative, respectively) integers and \( \mathbf{n} := \{ k \in \mathbb{N} \mid k \leq n \} \) for all \( n \in \mathbb{N}_0 \). Let \( \mathbb{N}^* \) be a free monoid over the alphabet \( \mathbb{N} \). We write \( q \cdot r \) for the concatenation product of \( q, r \in \mathbb{N}^* \) in order to avoid confusion with the ordinary product in \( \mathbb{N} \). Accordingly, we denote by \( d^k \) the \( k \)-th power of a letter \( d \in \mathbb{N} \) in \( \mathbb{N}^* \), for all \( k \in \mathbb{N}_0 \). If \( n \in \mathbb{N} \) and \( q = q_1, \ldots, q_k \in \mathbb{N}^* \) such that
Let $K$ be a field of characteristic 0 containing a primitive $n$-th root of unity $\zeta^n$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}_0$, we denote by $\text{Cl}_K(S_n)$ the ring of class functions of the symmetric group $S_n$. Let $C_q$ be the conjugacy class consisting of all permutations $\pi$ whose cycle partition $z(\pi)$ is a rearrangement of $q$, for all $q \in \mathbb{N}^*$. Let $\text{ch}_q \in \text{Cl}_K(S_n)$ such that $(\chi, \text{ch}_q)_{S_n} = \chi(C_q)$ is the value of $\chi$ on any element $\pi \in C_q$ for all $\chi \in \text{Cl}_K(S_n)$. Then, up to a certain factor, $\text{ch}_q$ is the characteristic function of $C_q$ in $\text{Cl}_K(S_n)$, and we have $C_q = C_r$ and $\text{ch}_q = \text{ch}_r$ whenever $q$ is a rearrangement of $r$, for all $q, r \in \mathbb{N}^*$. The outer product $\cdot$ on the direct sum $\text{Cl} := \bigoplus_{n \in \mathbb{N}_0} \text{Cl}_K(S_n)$ may now be defined by

\begin{equation}
\text{ch}_q \cdot \text{ch}_r := \text{ch}_{q \cdot r}
\end{equation}
for all $q, r \in \mathbb{N}^*$. It corresponds via Frobenius’ characteristic mapping to the ordinary multiplication of symmetric functions.

Our starting point is the following part of [12, Theorem 8.23], which already occurs in [16, Section 8].

**Lemma 2.1.** Let $n \in \mathbb{N}$ and $q \vdash n$. Denote by $a_i$ the multiplicity of the letter $i$ in $q$, for all $i \in \mathbb{N}$. Then we have $\lambda_q = \lambda_{n^a} \cdot \ldots \cdot \lambda_1^a$.

Hence, with $\zeta^p$ denoting the irreducible character of $S_n$ corresponding to $p$ for $p \vdash n$, the problem of describing the multiplicities $a_q \cdot p \vdash S_n$ may be reduced to the case that $q$ is of block type, that is, $q = d^k$ is the $k$-th power of a single letter $d$. Indeed, for partitions $q = q_1 \ldots q_k \vdash n$, $r = r_1 \ldots r_l \vdash y$ such that $q_i > r_i$ and $x + y = n$, we have

\begin{equation}
(\lambda_q \cdot \zeta^p)_{S_n} = (\lambda_q \cdot \lambda_r \cdot \zeta^p)_{S_n} = \sum_{x+y} c_{i,j}^p a_{q,i} a_{r,j}
\end{equation}
by Lemma 2.1, where $c_{i,j}^p = (\zeta^i \cdot \zeta^j, \zeta^p)_{S_n}$ is the well-known Littlewood-Richardson coefficient.

For all $n, m \in \mathbb{N}_0$, $\psi \in S_n$ and $\sigma \in S_m$, we define $\psi \# \sigma \in S_{n+m}$ by

\[ i(\psi \# \sigma) := \begin{cases} i \psi & i \leq n; \\ (i - n)\sigma + n & i > n \end{cases} \]

for all $i \in n + m$. Furthermore, for $d, k \in \mathbb{N}$, $n := dk$ and $\pi \in S_k$, we define $\pi^{(d_i)} \in S_n$ by

\[ (dj - i)\pi^{(d_i)} := d(j\pi) - i \]
for all \( j \in \mathcal{J} \), \( i \in \mathcal{I} \cup \{0\} \). That is, \( \pi^{(d^t)} \) is permuting the \( k \) successive blocks of length \( d \) in \( \mathcal{J} \) according to \( \pi \). Now let \( \tau_d := (1, \ldots, d) \in S_d \) be the standard cycle of length \( d \) in \( S_d \) and put

\[
\sigma_{d^t} \equiv \tau_d \# \cdots \# \tau_d \in C_{d^t} \subseteq S_n.
\]

Then the centralizer of \( \sigma_{d^t} \) in \( S_n \) is a wreath product of the cyclic group generated by \( \tau_d \) with \( S_k \) and may be described as

\[
C_{d^t} := C_{S_k}(\sigma_{d^t}) = \left\{ \pi^{(d^t)}(\tau_{d}^{i_1} \# \cdots \# \tau_{d}^{i_k}) \mid \pi \in S_k, i_1, \ldots, i_k \in \mathcal{I} \right\}.
\]

([5, Section 4.1]). With these notations, the remaining part of Theorem 8.23 in [12], transferred to Cl, reads as follows.

**Theorem 2.2.** Let \( d, k \in \mathbb{N} \) and \( n := dk \). Then

\[
\psi_{d^t} : C_{d^t} \longrightarrow K, \quad \pi^{(d^t)}(\tau_{d}^{i_1} \# \cdots \# \tau_{d}^{i_k}) \longmapsto e^{-(i_1 + \cdots + i_k)}
\]

is a linear representation of \( C_{d^t} \), and \( (\psi_{d^t})^{S_k} = \lambda_{d^t} \).

### 3. Multiplicities

In order to state our main result (Theorem 3.1), we need the notion of a standard Young tableau and its multi major index corresponding to a composition. Let \( n \in \mathbb{N} \) and \( p = p_1, \ldots, p_l \vdash n \). The frame \( R(p) := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i \in \mathcal{I}, j \in \mathcal{J} \} \) corresponding to \( p \) may be visualized by its Ferrers diagram, an array of boxes with \( p_1 \) boxes in the first (top) row, \( p_2 \) boxes in the second row and so on. For example, we have

\[ R(3,2) \sim \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array} \]

The images \( 1\pi, \ldots, n\pi \) of any permutation \( \pi \in S_n \) may be entered into \( R(p) \) row by row, starting at bottom left and ending at top right. Let SYT\(^p\) be the set of all permutations which are increasing in rows (from left to right) and columns (downwards) when entered into \( R(p) \) in this way. The elements of SYT\(^p\) are called *standard Young tableaux* of shape \( p \). In the above example, the elements of SYT\(^{3,2}\), entered into \( R(3,2) \), are

\[
\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array}, \quad \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array}, \quad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \end{array}, \quad \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array}, \quad \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \end{array}
\]
Accordingly, we obtain

\[
\text{SYT}^3 = \left\{ \begin{bmatrix} 12345 \\ 45123 \\ 25134 \\ 24135 \\ 34125 \end{bmatrix}, \begin{bmatrix} 12345 \\ 35124 \\ 25134 \\ 24135 \\ 34125 \end{bmatrix}, \begin{bmatrix} 12345 \\ 25134 \\ 24135 \\ 34125 \end{bmatrix}, \begin{bmatrix} 12345 \\ 34125 \end{bmatrix} \right\} \subseteq S_5.
\]

For all \( \pi \in S_n \), \( D(\pi) := \{ i \in \{ n - 1 \} \mid i \pi > (i + 1)\pi \} \) is called the descent set of \( \pi \).

Let \( q = q_1, \ldots, q_k \models n \) and put \( s_j := q_1 + \cdots + q_j \) for all \( j \in \mathbb{N} \cup \{0\} \). Then the multi major index of \( \pi \) corresponding to \( q \) is defined as

\[
\text{maj}_q \pi := m_1, \ldots, m_k \in \mathbb{N}^*,
\]

where

\[
m_j := \sum_{s_{j-1} < i \leq s_j} (i - s_{j-1})\]

for all \( j \in \mathbb{N} \). For \( q = n \), we obtain the ordinary major index \( \text{maj} \pi := \text{maj}_n \pi \) of \( \pi \).

If, additionally, \( r = r_1, \ldots, r_k \in \mathbb{N}^* \), we define

\[
\text{SYT}^r_p := \left\{ \pi \in \text{SYT}^p \mid \forall j \in \mathbb{N} \cdot (\text{maj}_q(\pi^{-1}))_j \equiv r_j \mod q_j \right\}.
\]

Here \( (\text{maj}_q(\pi^{-1}))_j \) always denotes the \( j \)-th letter of \( \text{maj}_q(\pi^{-1}) \), for all \( j \in \mathbb{N} \). For arbitrary \( r = r_1, \ldots, r_k \), \( q = q_1, \ldots, q_k \in \mathbb{N}^* \) we write \( r \mid q \) if and only if \( l = k \) and \( r_i \) is a divisor of \( q_i \) for all \( i \in \mathbb{N} \). In this case, we define furthermore the following extension of the number theoretic M"obius function \( \mu \):

\[
\mu(q/r) := \prod_{i=1}^{\lfloor q \rfloor} \mu(q_i/r_i).
\]

Finally, for \( k \in \mathbb{N} \) and \( r = r_1, \ldots, r_k \in \mathbb{N}^* \), we put \( k \ast r := (kr_1), \ldots, (kr_k) \).

**Main Theorem 3.1.** Let \( d, n \in \mathbb{N} \) such that \( dk = n \). Let \( p \vdash n \). Then we have

\[
(\lambda_{d^k}, \pi^p)_{S_k} = \frac{1}{k!} \sum_{q \vdash k} |C_q| \sum_{r \vdash q} \mu(q/r) \text{SYT}^r_p.
\]

The proof will be given in Section 5. A description of the multiplicity \( (\lambda_q, \pi^p)_{S_k} \) for arbitrary \( q \vdash n \) may be obtained from Theorem 3.1 via (3). For \( k \leq 3 \), we obtain the following specializations of Theorem 3.1, the first of which is due to Kraskiewicz and Weyman (see the Remark at the end of this section).

**Corollary 3.2.** Let \( d \in \mathbb{N} \).

(a) For all \( p \vdash d \), we have \( (\lambda_d, \pi^p)_{S_d} = \text{SYT}^p_{d,1} \).

(b) For all \( p \vdash 2d \), we have \( (\lambda_{d^2}, \pi^p)_{S_{d^2}} = 1/2(\text{SYT}^p_{d^2,1,1} + \text{SYT}^p_{d^2,2} - \text{SYT}^p_{d^2,1,1}) \).
(c) For all \( p \vdash 3d \), we have

\[
(\lambda_{d,d,d}, \xi^p)_{S_n} = \frac{1}{6} \left( \text{syt}^p_{d,d,d,1,1,1} + 3(\text{syt}^p_{2d,d,2,1} - \text{syt}^p_{2d,d,1,1}) + 2(\text{syt}^p_{3d,3} - \text{syt}^p_{3d,1}) \right).
\]

We will illustrate Corollary 3.2 in the case of \( p = 2.2.2 \). The standard Young tableaux \( \pi \) of shape \( p \) are listed in Table 1 together with their multi major indices in question. The descents of \( \pi^{-1} \) are underlined in each case.

By Corollary 3.2, we obtain \((\lambda_{d}, \xi^{2.2.2})_{S_n} = 0\) and furthermore

\[
(\lambda_{3,3}, \xi^{2.2.2})_{S_n} = \frac{1}{2} (1 + 1 - 0) = 1
\]

and

\[
(\lambda_{2.2.2}, \xi^{2.2.2})_{S_n} = \frac{1}{6} (1 + 3(0 - 1) + 2(1 - 0)) = 0.
\]

For \( p \vdash d \in \mathbb{N} \) and \( \pi \in \text{SYT}^p \), note that \( i \in d-1 \) is a descent of \( \pi^{-1} \) if and only if \( i \) stands strictly above \( i+1 \) in \( \pi \), entered into \( R(p) \). Hence Corollary 3.2 (a) indeed coincides with the original result of Kraskiewicz and Weyman on the Lie character \( \lambda_d \) ([9]).
### 4. Noncommutative character theory

Let \( n \in \mathbb{N} \). The **descent algebra** \( D_n \) is defined as the linear span of the elements
\[
\delta^D := \sum_{\pi \in S_n} D(\pi) = D \begin{pmatrix} 1 & \cdots & n \end{pmatrix} \text{ in } KS_n.
\]
Due to Solomon ([15]), \( D_n \) is a subalgebra of \( KS_n \), and there exists a certain epimorphism of algebras \( c_n : D_n \to Cl_K(S_n) \), for all \( n \). The direct sum \( KS := \bigoplus_{n \in \mathbb{N}} KS_n \) is a graded algebra with respect to the convolution product \( \ast \) (see [6, 1.3] for a combinatorial description), and \( D := \bigoplus_{n \in \mathbb{N}} D_n \) is a \( \ast \)-subalgebra of \( KS \) (see [12]). In [6], a (noncommutative) \( \ast \)-subalgebra \( B \) of \( KS \) and a \( \ast \)-homomorphism \( c_B : B \to Cl \) are introduced such that \( D \subseteq B \) and \( c_B \) is a subalgebra for all \( n \). Furthermore, a (bilinear) scalar product \( (\cdot, \cdot) \) on \( KS \) is defined by
\[
(\pi, \sigma) := \begin{cases} 1 & \pi = \sigma^{-1}; \\ 0 & \pi \neq \sigma^{-1} \end{cases}
\]
for all permutations \( \pi, \sigma \), and it is shown that
\[
(\varphi, \psi) = (c(\varphi), c(\psi))_S
\]
for all \( \varphi, \psi \in B \), where the scalar product on the right hand side is the canonical orthogonal extension of the ordinary scalar products \( (\cdot, \cdot)_S \) on \( Cl_K(S_n) \), \( n \in \mathbb{N} \). For any partition \( p \in \mathbb{N}^* \), \( Z^p := \sum_{\pi \in SYT_p} \pi \) is an element of \( B \) such that
\[
c(Z^p) = \zeta^p
\]
is the irreducible character of \( S_n \) corresponding to \( p \). For example, for \( p = 3.2 \), we obtain \( Z^{3.2} = \left( \begin{array}{c} 12345 \\ 45123 \end{array} \right) + \left( \begin{array}{c} 12345 \\ 35124 \end{array} \right) + \left( \begin{array}{c} 12345 \\ 21345 \end{array} \right) + \left( \begin{array}{c} 12345 \\ 25134 \end{array} \right) \). These results provide the following general concept for describing multiplicities: Given an arbitrary character \( \chi \in Cl_K(S_n) \), any inverse image \( \varphi \in B \) of \( \chi \) under \( c \) may be understood as a noncommutative character corresponding to \( \chi \). By (8) and (9), for each such \( \varphi \), it follows that
\[
(\chi, \zeta^p)_S = (c(\varphi), c(Z^p))_S = (\varphi, Z^p).
\]
The right-hand side of (10) gives different combinatorial descriptions of the multiplicity on the left-hand side, according to the choice of \( \varphi \), simply by the definition of \( Z^p \) and the scalar product on \( B \).

### 5. Klyachkos’s idempotent and Ramanujan sums

In the sequel, following the concept described in Section 4, an inverse image of \( \lambda_{d^k} \) under \( c \) in \( D \) is constructed. It leads to a short proof of our main result Theorem 3.1, by means of (10).
Let \( n \in \mathbb{N} \). We put \( \kappa_n(x) := \sum_{\pi \in \mathcal{C}_n} x^{\text{maj}\pi} \) (\( x \) a variable) and

\[
M_{n,i} := \sum_{\pi \in \mathcal{C}_n \mod n} \pi \in \mathcal{D}_n
\]

for all \( i \in \mathbb{N}_0 \). Then, up to the factor \( 1/n \), \( \kappa_n(e_n) = \sum_{i=1}^n e_n^i M_{n,i} \in \mathcal{D}_n \) is a Lie idempotent, that is, \( \kappa_n^2 = n \kappa_n \) and \( L_n(V) = \kappa_n T_n(V) \). This remarkable result is due to Klyachko ([8]).

**Lemma 5.1.** Let \( n, i \in \mathbb{N} \) and \( d \) be the order of \( e_n^i \). Then we have

\[
\kappa_n(e_n^i) = \kappa_d(e_n^i) \cdot \cdots \cdot \kappa_d(e_n^i).
\]

In particular, \( c(\kappa_n(e_n^i)) = c(\kappa_n) \).\[ /text]

The main part of the preceding lemma is a special case of [10, Proposition 4.1], while the additional claim on the \( c \)-image follows from [7, Proposition 1]. For \( n, m \in \mathbb{N} \), we denote by \( \gcd(n, m) \) the greatest common divisor of \( n \) and \( m \).

**Corollary 5.2.** Let \( n \in \mathbb{N} \) and \( i, j \in \mathbb{N}_0 \) such that \( \gcd(i, n) = \gcd(j, n) \). Then

\[
c(M_{n,i}) = c(M_{n,j}).
\]

**Proof.** As \( \gcd(i, n) = \gcd(j, n) \), we can find an integer \( m \in \mathbb{N} \) such that \( i \equiv jm \) modulo \( n \) and \( \gcd(m, n) = 1 \). For all \( k \in \mathbb{N} \), we have \( \gcd(km, n) = \gcd(k, n) \) and hence \( c(\kappa_n(e_n^k)) = c(\kappa_n(e_n^{mk})) \), by Lemma 5.1. It follows that

\[
nc(M_{n,i}) = c \left( \sum_{i=1}^n \sum_{k=1}^n (e_n^{i-k})^k M_{n,i} \right) = c \left( \sum_{k=1}^n e_n^{-ik} \kappa_n(e_n^k) \right)
\]

\[
= c \left( \sum_{k=1}^n e_n^{-ik} \kappa_n(e_n^{mk}) \right) = c \left( \sum_{i=1}^n \sum_{k=1}^n (e_n^{i-k})^k M_{n,i} \right)
\]

\[
= c \left( \sum_{i=1}^n \sum_{k=1}^n (e_n^{i-k})^k M_{n,i} \right) = nc(M_{n,i}). \quad \square
\]

Let \( n, m \in \mathbb{N} \). The Ramanujan sum corresponding to \( n \) and \( m \) is defined by

\[
\varrho(n, m) := \sum e_m,
\]

where the sum is taken over all primitive \( n \)-th roots of unity \( \epsilon \). In the particular case of \( m = 1 \) (\( m = n \), respectively), \( \varrho(n, m) \) yields the Möbius function \( \mu(n) = \varrho(n, 1) \).
(Euler’s function $\varphi(n) = \varphi(n, n)$, respectively). We write $x \mid m$, if $x \in \mathbb{N}$ is a divisor of $m$, and put

$$ R(n, m) := \sum_{x \mid m} \varphi(n, x)\varphi(m/x, 1). $$

Now, for all $d, k \in \mathbb{N}$ and $p = p_1, \ldots, p_l \in \mathbb{N}^+$, let

$$ M_d(k) := \sum_{y \mid dk} R(dk/y, d)M_{dk,y} $$

and

$$ M_d(p) := M_d(p_1) \cdot \cdots \cdot M_d(p_l). $$

Note that $M_d(p) \in \mathcal{D}$, as $\mathcal{D}$ is closed under the convolution product.

**Lemma 5.3.** For all $d, k \in \mathbb{N}$, we have

$$ \lambda_{d^k} = c \left( \frac{1}{k!} \sum_{\pi \in S_k} \frac{1}{d^{\ell(\pi)}} M_d(z(\pi)) \right). $$

(Recall that $z(\pi)$ denotes the cycle partition of $\pi$ for any permutation $\pi$.)

**Proof.** We write

$$ z(\pi; i_1, \ldots, i_k) := z(\pi^{[d^k]}(\tau_1^{i_1} \cdots \tau_k^{i_k})) $$

for all $\pi \in S_k, i_1, \ldots, i_k \in d - 1 \cup \{0\}$. By Theorem 2.2, we then have

$$ \lambda_{d^k} = \frac{1}{|C^{d^k}|} \sum_{\varphi \in C^{d^k}} \left( \sum_{\varphi \in C^{d^k}} \psi_{d^k}(\varphi) \right) \text{ch}_q $$

$$ = \frac{1}{k!} \sum_{\pi \in S_k} \frac{1}{d^\ell} \sum_{i_1, \ldots, i_k = 0}^{d-1} e_d^{-\sum_{i_1}^{i_k}} \text{ch}_{z(\pi; i_1, \ldots, i_k)}. $$

By induction on the number $z = |z(\pi)|$ of cycles in $\pi \in S_k$, we show that

$$ \frac{1}{d^k} \sum_{i_1, \ldots, i_k = 0}^{d-1} e_d^{-\sum_{i_1}^{i_k}} \text{ch}_{z(\pi; i_1, \ldots, i_k)} = c \left( \frac{1}{d^k} M_d(z(\pi)) \right), $$

which implies our claim. We will use some basic facts about cycle partitions of elements of $C^{d^k}$ which can be found in [5, 4.2]. Let $z = 1$. Then $\pi \in S_k$ is a long
cycle. Putting \( \eta := \varepsilon_{kd} \) and applying [5, 4.2.17], Lemma 5.1 and Corollary 5.2, we obtain

\[
\frac{1}{d^k} \sum_{i_1, \ldots, i_k = 0}^{d-1} \varepsilon_d^{\sum_i i_i} \chi_{i}(\sigma_i^{j_1, \ldots, j_k}) = \frac{1}{d} \sum_{x/d} \varepsilon_d^{i} \chi_{x}(\tau_{i}^{j_2}) = \frac{1}{d} \sum_{x/d} \varepsilon(d/x, 1) \chi_{x}(\tau_{i}^{j_2}) = c \left( \frac{1}{d} \sum_{x/d} \varepsilon(d/x, 1) \chi_{x}(\tau_{i}^{j_2}) \right) = c \left( \frac{1}{d} \sum_{y/d} M_{dk}^{(y)} R(dk/y, d) \right) = c(M_d(k)/d).
\]

Now let \( z > 1 \), say, \( \pi = \tilde{\pi} \sigma \) for a cycle \( \sigma \) of length \( l \) in \( \pi \). Then we have, by [5, 4.2.19], (2) and our induction hypothesis,

\[
\frac{1}{d^{k-1}} \sum_{i_1, \ldots, i_{k-1} = 0}^{d-1} \varepsilon_d^{\sum_i i_i} \chi_{i}(\sigma_i^{j_1, \ldots, j_{k-1}}) = \left( \frac{1}{d^{k-1}} \sum_{i_1, \ldots, i_{k-1} = 0}^{d-1} \varepsilon_d^{\sum_i i_i} \chi_{i}(\sigma_i^{j_1, \ldots, j_{k-1}}) \right) \ast \left( \frac{1}{d} \sum_{y/d} M_{dk}^{(y)} (z(\tilde{\pi})) \right) = c \left( \frac{1}{d} M_d(z(\pi)) \right).
\]

This completes the proof of (\*). \( \square \)

The inverse image of \( \lambda_d \) under \( c \) constructed in the preceding lemma may be simplified by means of a short analysis of the numbers \( R(n, m) \). This will be done in three steps.

**Proposition 5.4.** Let \( n_1, n_2, m_1, m_2 \in \mathbb{N} \) such that

\[
\gcd(n_1, n_2) = \gcd(m_1, m_2) = \gcd(n_1, m_1) = \gcd(n_2, m_1) = 1.
\]

Then we have \( R(n_1n_2, m_1m_2) = R(n_1, m_1)R(n_2, m_2) \).

**Proof.** By [4, Theorem 67], the Ramanujan sums have the following factorizing property: \( \varphi(a_1a_2, b) = \varphi(a_1, b)\varphi(a_2, b) \) for all \( a_1, a_2, b \in \mathbb{N} \) such that \( \gcd(a_1, a_2) = 1 \). Furthermore, we have \( \varphi(a, b_1b_2) = \varphi(a, b_1) \) for all \( a, b_1, b_2 \in \mathbb{N} \) such that \( (a, b_2) = 1 \),
as in this case taking the \( b_2 \)-th power induces an automorphism of the group of \( a \)-th roots of unity. These two observations imply that

\[
R(n_1, n_2, m_1, m_2) = \sum_{x_1 \mid n_1} \sum_{x_2 \mid n_2} \phi(n_1, x_1, x_2) \phi\left(\frac{m_1}{x_1}, \frac{m_2}{x_2}, 1\right)
\]

\[
= \sum_{x_1 \mid n_1} \sum_{x_2 \mid n_2} \phi(n_1, x_1) \phi(n_2, x_1, x_2) \phi\left(\frac{m_1}{x_1}, 1\right) \phi\left(\frac{m_2}{x_2}, 1\right)
\]

\[
= \sum_{x_1 \mid n_1} \phi(n_1, x_1) \phi\left(\frac{m_1}{x_1}, 1\right) \sum_{x_2 \mid n_2} \phi(n_2, x_2) \phi\left(\frac{m_2}{x_2}, 1\right)
\]

\[
= R(n_1, m_1) R(n_2, m_2). \tag*{\square}
\]

Let \( \mathbb{P} \) be the set of all prime numbers.

**Proposition 5.5.** For all \( a, b \in \mathbb{N}_0 \) and \( p \in \mathbb{P} \), we have

\[
R(p^a, p^b) = \begin{cases} 
\mu(p^{a-b}) p^b & b \leq a; \\
0 & b > a.
\end{cases}
\]

**Proof.** For all \( n, m \in \mathbb{N} \), the Ramanujan sum corresponding to \( n \) and \( m \) may be expressed in terms of the Möbius and the Euler function as follows:

\[
\phi(n, m) = \mu(n / \gcd(n, m)) \frac{\phi(n)}{\phi(n / \gcd(n, m))}
\]

([4, Theorem 272]). Let \( c := \min\{a, b\} \) and \( d := \min\{a, b - 1\} \). Then

\[
R(p^a, p^b) = \sum_{i=0}^{b} \phi(p^a, p^i) \phi(p^{b-i}, 1)
\]

\[
= \phi(p^a, p^b) - \phi(p^a, p^{b-1})
\]

\[
= \mu(p^{a-c}) \frac{\phi(p^a)}{\phi(p^{a-c})} - \mu(p^{a-d}) \frac{\phi(p^a)}{\phi(p^{a-d})}
\]

and hence \( R(p^a, p^b) = 0 \) for \( b > a \), as \( c = d = a \) in this case. Let \( b \leq a \). Then we have \( c = b \) and \( d = b - 1 \), that is,

\[
R(p^a, p^b) = \mu(p^{a-b}) \frac{\phi(p^a)}{\phi(p^{a-b})} - \mu(p^{a-b+1}) \frac{\phi(p^a)}{\phi(p^{a-b+1})}.
\]

For \( b < a - 1 \), this shows \( R(p^a, p^b) = 0 \) as asserted. For \( b = a - 1 \) it follows that \( R(p^a, p^b) = -\frac{\phi(p^{b+1})}{\phi(p)} = -p^b \), while, for \( b = a \), we may conclude that \( R(p^a, p^b) = \phi(p^a) - \phi(p^b) / \phi(p) = p^b \).

\tag*{\square}
**Lemma 5.6.** For all \( n, m \in \mathbb{N} \), we have
\[
R(n, m) = \begin{cases} 
\mu(n/m)m & m \mid n; \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Choose \( a_p, b_p \in \mathbb{N}_0 \) for all \( p \in \mathbb{P} \) such that \( n = \prod_{p \in \mathbb{P}} p^{a_p} \) and \( m = \prod_{p \in \mathbb{P}} p^{b_p} \).
Applying Propositions 5.4 and 5.5 we obtain
\[
R(n, m) = \prod_{p \in \mathbb{P}} R(p^{a_p}, p^{b_p})
= \begin{cases} 
\prod_{p \in \mathbb{P}} \mu(p^{a_p-b_p})p^{b_p} & \forall p \in \mathbb{P} : b_p \geq a_p; \\
0 & \text{otherwise}
\end{cases}
= \begin{cases} 
\mu(n/m)m & m \mid n; \\
0 & \text{otherwise}.
\end{cases}
\]

**Corollary 5.7.** Let \( d, k \in \mathbb{N} \). Then \( M_d(k) = d \sum_{y \mid k} \mu(k/y)M_{dk/y} \).

**Proof.** Let \( y \) be a divisor of \( dk \). Then Lemma 5.6 implies that
\[
R(dk/y, d) = \begin{cases} 
\mu(dk/dy)d & d \mid dk/y; \\
0 & \text{otherwise}
\end{cases} = \begin{cases} 
\mu(k/y)d & y \mid k; \\
0 & \text{otherwise}.
\end{cases}
\]

We are now in a position to give the proof of the Main Theorem 3.1.

**Proof of the Main Theorem 3.1.** By Lemma 5.3 and (10), we have
\[
(\lambda_{d^k}, \xi^p)^{S_k} = \frac{1}{k!} \sum_{z \in S_k} \frac{1}{d^{[z(\pi)]}} (M_d(z(\pi)), Z^p).
\]
But, for \( \pi \in S_k \) and \( q = q_1, \ldots, q_k := z(\pi) \), we may conclude from Corollary 5.7 that
\[
\frac{1}{d^{[z(\pi)]}} (M_d(z(\pi)), Z^p) = \frac{1}{d^k} (M_d(q_1) \cdot \cdots \cdot M_d(q_k), Z^p)
= \sum_{r_1 | q_1} \cdots \sum_{r_k | q_k} \mu(q_1/r_1) \cdots \mu(q_k/r_k) (M_{d_{q_1, r_1}} \cdot \cdots \cdot M_{d_{q_k, r_k}}, Z^p)
= \sum_{r | q} \mu(q/r) (M_{d_{q_1, r_1}} \cdot \cdots \cdot M_{d_{q_k, r_k}}, Z^p).
\]
This completes the proof, as \((M_{d_{q_1, r_1}} \cdot \cdots \cdot M_{d_{q_k, r_k}}, Z^p) = \text{syt}_{q,r}^p\) for all \( r \mid q \), simply by definition of the scalar product \((\cdot, \cdot)\) and the convolution product \(\bullet\) in [6, 1.3].
References


Mathematical Institute
24–29 St Giles
Oxford OX1 3LB
UK
e-mail: schocker@maths.ox.ac.uk