

ON \mathfrak{F} -HYPEREXCENTRIC MODULES FOR LIE ALGEBRAS

DONALD W. BARNES

(Received 23 November 2001; revised 4 February 2002)

Communicated by R. B. Howlett

Abstract

Let \mathfrak{F} be a saturated formation of soluble Lie algebras over the field F , and let $L \in \mathfrak{F}$. Let V and W be \mathfrak{F} -hypercentral and \mathfrak{F} -hyperexcentric L -modules respectively. Then $V \otimes_F W$ and $\text{Hom}_F(V, W)$ are \mathfrak{F} -hyperexcentric and $H^n(L, W) = 0$ for all n .

2000 *Mathematics subject classification*: primary 17B30, 17B56; secondary 17B50, 17B55.

Keywords and phrases: Soluble Lie algebras, cohomology.

1. Introduction

Let \mathfrak{F} be a saturated formation of finite-dimensional soluble Lie algebras over the field F . Let $L \in \mathfrak{F}$ and let W be an \mathfrak{F} -excentric irreducible L -module. Results in Barnes and Gastineau-Hills [4] imply that $H^n(L, W) = 0$ for $n \leq 2$, and $H^n(L, W) = 0$ for all n was proved for some special cases, suggesting that this might be true in general. This was proved in Barnes [3] for fields F of characteristic 0. The proof involved a description of the saturated formations over an arbitrary field of characteristic 0. Over a field of non-zero characteristic, the saturated formations are much more complicated and no useful description is available. In this paper, we give a proof independent of the characteristic of the field. All algebras and modules considered are assumed finite-dimensional over F .

An irreducible L -module V is called \mathfrak{F} -central if the split extension of V by $L/\mathcal{C}_L(V)$ is in \mathfrak{F} and \mathfrak{F} -excentric otherwise. An L -module V is called \mathfrak{F} -hypercentral if every composition factor of V is \mathfrak{F} -central and is called \mathfrak{F} -hyperexcentric if ev-

This work was done while the author was an Honorary Associate of the School of Mathematics and Statistics, University of Sydney.

© 2003 Australian Mathematical Society 1446-8107/03 \$A2.00 + 0.00

ery composition factor of V is \mathfrak{F} -excentric. We need the following theorems from Barnes [2].

THEOREM 1.1 ([2, Theorem 4.4]). *Suppose $L \in \mathfrak{F}$ and let V be an L -module. Then V is the direct sum of an \mathfrak{F} -hypercentral L -module and an \mathfrak{F} -hyperexcentric L -module.*

THEOREM 1.2 ([2, Theorem 2.1]). *Let V and W be \mathfrak{F} -hypercentral L -modules. Then $V \otimes_F W$ and $\text{Hom}_F(V, W)$ are \mathfrak{F} -hypercentral.*

Results showing that $H^n(L, V) = 0$ for \mathfrak{F} -excentric irreducible L -modules V are easily extended to \mathfrak{F} -hyperexcentric modules by using the cohomology exact sequence and induction over the composition length of the module.

2. \mathfrak{F} -hyperexcentric modules

In this section, we obtain a cohomological characterisation of \mathfrak{F} -hyperexcentric L -modules. The characterisation needs to use other algebras besides the algebra L from which we start.

DEFINITION 2.1. Suppose $L \in \mathfrak{F}$. The *cone of L* in \mathfrak{F} is the class (\mathfrak{F}/L) of all pairs (M, ϵ) where $M \in \mathfrak{F}$ and $\epsilon : M \rightarrow L$ is an epimorphism. We usually omit ϵ from the notation, writing simply $M \in (\mathfrak{F}/L)$.

Any L -module V is an M -module via ϵ for any $M \in (\mathfrak{F}/L)$. Then V is \mathfrak{F} -hypercentral as M -module if and only if it is \mathfrak{F} -hypercentral as L -module. It follows that if V is an \mathfrak{F} -hyperexcentric L -module, then $H^n(M, V) = 0$ for all $M \in (\mathfrak{F}/L)$ and $n \leq 2$. We would like a converse to this.

THEOREM 2.2. *Let \mathfrak{F} be a saturated formation and let $L \in \mathfrak{F}$. Suppose V is an L -module such that for all $M \in (\mathfrak{F}/L)$, $H^1(M, V) = 0$. Then V is \mathfrak{F} -hyperexcentric.*

PROOF. V is the direct sum of an \mathfrak{F} -hypercentral module and an \mathfrak{F} -hyperexcentric module. Thus we may suppose without loss of generality, that V is \mathfrak{F} -hypercentral, and we then have to prove $V = 0$. Suppose $V \neq 0$ and let W be a minimal submodule of V . We form the direct sum A of sufficiently many copies of W to ensure that $\dim \text{Hom}_L(A, V) > \dim H^2(L, V)$, and construct the split extension M of A by L . As W is \mathfrak{F} -central, $M \in (\mathfrak{F}/L)$. We use the Hochschild-Serre spectral sequence to calculate $H^1(M, V)$. We have

$$E_2^{20} = H^2(M/A, V^A) = H^2(L, V)$$

and

$$E_2^{01} = H^0(M/A, H^1(A, V)) = \text{Hom}_F(A, V)^L = \text{Hom}_L(A, V).$$

Thus $\dim d_2^{01}(E_2^{01}) \leq \dim H^2(L, V) < \dim E_2^{01}$, so $E_3^{01} = \ker d_2^{01} \neq 0$ and so $H^1(M, V) \neq 0$ contrary to assumption. \square

THEOREM 2.3. *Let \mathfrak{F} be a saturated formation and let $L \in \mathfrak{F}$. Suppose V is an \mathfrak{F} -hypercentral L -module and let W be an \mathfrak{F} -hyperexcentric L -module. Then $V \otimes_F W$ and $\text{Hom}_F(V, W)$ are \mathfrak{F} -hyperexcentric.*

PROOF. Let $M \in (\mathfrak{F}/L)$. Then V and W are \mathfrak{F} -hypercentral and \mathfrak{F} -hyperexcentric respectively as M -modules, and every M -module extension X of W by V splits. Thus $H^1(M, \text{Hom}_F(V, W)) = 0$. By Theorem 2.2, $\text{Hom}_F(V, W)$ is \mathfrak{F} -hyperexcentric. By Theorem 1.2, the dual module $V^* = \text{Hom}_F(V, F)$ is \mathfrak{F} -hypercentral. As

$$V \otimes_F W \simeq V^{**} \otimes_F W \simeq \text{Hom}_F(V^*, W),$$

the result follows. \square

This suggests that we could have some sort of \mathbb{Z}_2 -grading on the class of all L -modules. However, the tensor product of two \mathfrak{F} -hyperexcentric modules need not be \mathfrak{F} -hypercentral. Anything can happen as is shown by the following examples. Here, \mathfrak{N} denotes the saturated formation of all nilpotent algebras.

EXAMPLE 2.4. Suppose the characteristic of F is not 2. Let $L = \langle e \rangle$ be the 1-dimensional algebra, and let $V = \langle v \rangle$ and $W = \langle w \rangle$ be the modules with action given by $ev = v$ and $ew = w$. Then V and W are \mathfrak{N} -excentric and $V \otimes_F W$ is \mathfrak{N} -excentric.

EXAMPLE 2.5. Let $L = \langle e \rangle$ be the 1-dimensional algebra, and let $V = \langle v \rangle$ and $W = \langle w \rangle$ be the modules with action given by $ev = v$ and $ew = -w$. Then V and W are \mathfrak{N} -excentric and $V \otimes_F W$ is \mathfrak{N} -central.

EXAMPLE 2.6. Suppose the characteristic of F is not 2. Let $i \in \bar{F}$ have minimum polynomial $x^2 + 1$. Let $L = \langle e \rangle$ be the 1-dimensional algebra, and let V and W be 2-dimensional modules with the action given by the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The eigenvalues of the action of e on $V \otimes_F W$ are the sums of the eigenvalues on V and W , thus $2i, 0, 0, -2i$. Thus $V \otimes_F W$ is the direct sum of a 2-dimensional module on which the action is trivial, and a 2-dimensional module on which the action is given by the matrix $2A$. It is thus the sum of an \mathfrak{N} -hypercentral and an \mathfrak{N} -excentric module.

3. Cohomology of \mathfrak{F} -hyperexcentric modules

We can now prove the desired theorem on the cohomology of \mathfrak{F} -hyperexcentric modules.

THEOREM 3.1. *Let \mathfrak{F} be a saturated formation and let $L \in \mathfrak{F}$. Let V be an \mathfrak{F} -hyperexcentric L -module. Then $H^n(L, V) = 0$ for all n .*

PROOF. By the cohomology exact sequence for a submodule W

$$\cdots \rightarrow H^n(L, W) \rightarrow H^n(L, V) \rightarrow H^n(L, V/W) \rightarrow \cdots,$$

we need only consider the case in which V is irreducible. We use induction over $\dim L$. The result holds if $\dim L = 1$, so suppose $\dim L > 1$. Let A be a minimal ideal of L . We use the Hochschild-Serre spectral sequence. We have

$$E_2^{rs} = H^r(L/A, H^s(A, V)).$$

If A acts non-trivially on V , then $V^A = 0$ and $H^s(A, V) = 0$ for all s by Barnes [1, Theorem 1]. If on the other hand, A acts trivially on V , then $H^s(A, V) = \text{Hom}_F(\Lambda^s A, V)$. Now $\Lambda^s A$ is a submodule of the tensor power of A , so is \mathfrak{F} -hypercentral by Theorem 1.2. By Theorem 2.3, $\text{Hom}_F(\Lambda^s A, V)$ is \mathfrak{F} -hyperexcentric. By induction over $\dim L$, we have $H^r(L/A, H^s(A, V)) = 0$ for all r, s . In either case, we have $H^r(L/A, H^s(A, V)) = 0$ for all r, s . By the Hochschild-Serre spectral sequence, $H^n(L, V) = 0$ for all n . \square

References

- [1] D. W. Barnes, 'On the cohomology of soluble Lie algebras', *Math. Z.* **101** (1967), 343–349.
- [2] ———, 'On \mathfrak{F} -hypercentral modules for Lie algebras', *Arch. Math.* **30** (1978), 1–7.
- [3] ———, 'Saturated formations of soluble Lie algebras in characteristic 0', *Arch. Math.* **30** (1978), 477–480.
- [4] D. W. Barnes and H. M. Gastineau-Hills, 'On the theory of soluble Lie algebras', *Math. Z.* **106** (1968), 343–354.

1 Little Wonga Rd.
 Cremorne NSW 2090
 Australia
 e-mail: donb@netspace.net.au