MANIFOLDS THAT FAIL TO BE CO-DIMENSION 2 FIBRATORS NECESSARILY COVER THEMSELVES

YOUNG HO IM and YONGKUK KIM

(Received 17 July 2001; revised 18 January 2002)

Communicated by S. Gadde

Abstract

Let \( N \) be a closed s-Hopfian \( n \)-manifold with residually finite, torsion free \( \pi_1(N) \) and finite \( H_1(N) \). Suppose that either \( \pi_1(N) \) is finitely generated for all \( k \geq 2 \), or \( \pi_k(N) \cong 0 \) for \( 1 < k < n-1 \), or \( n \leq 4 \). We show that if \( N \) fails to be a co-dimension 2 fibration, then \( N \) cyclically covers itself, up to homotopy type.


Keywords and phrases: residually finite group; s-Hopfian manifold; approximate fibration.

1. Introduction

The advantage of approximate fibration is that on one hand there exists an exact homotopy sequence but on the other hand there are more such approximate fibrations available. (See [3, 4, 5] for the definition and usefulness of approximate fibrations.)

To detect approximate fibrations, Daverman introduced the concept of co-dimension 2 fibration as follows [7].

A closed \( n \)-manifold \( N^k \) is a co-dimension 2 fibration (respectively, a co-dimension 2 orientable fibration) if, whenever \( p : M \to B \) is a proper map from an arbitrary (respectively, orientable) \( (n+2) \)-manifold \( M \) to a 2-manifold \( B \) such that each \( p^{-1}(b) \) is shape equivalent to \( N \), then \( p : M \to B \) is an approximate fibration.

All closed s-Hopfian manifolds with either trivial fundamental group or Hopfian fundamental group and nonzero Euler characteristic or hyper-Hopfian fundamental group are known to be co-dimension 2 fibrators [7, 6, 16, 19, 18].
For the sake of simplicity, we say that a closed manifold $N$ satisfies (CP) if $N$ cyclically covers itself nontrivially, up to homotopy type, and say that a closed manifold $N$ is $(F)$ if $N$ fails to be a co-dimension 2 fibration. Not only the torus and the Klein bottle but also $S^1$-bundles satisfy (CP).

It is well known [7, Theorem 4.2] that if a closed manifold $N$ satisfies (CP), then $N$ is $(F)$. What can we say about the converse? Recently, Daverman [9] proves that the converse is not true in general, by showing that $S^3 \times L(p, q)$ fails to be a co-dimension 2 fibration but it cannot cover itself cyclically, where $L(p, q)$ is a Lens space.

It is natural to ask when the converse is true. A continuation of earlier investigations launched on [17], this paper adds evidence for a claim that the converse is true for many interesting manifolds. More precisely, we have the following

**THEOREM.** Suppose that a closed $s$-Hopfian $n$-manifold $N$ with residually finite, torsion free $\pi_1(N)$ and finite $H_1(N)$ is $(F)$. Then, $N$ satisfies (CP), provided either

1. $\pi_k(N)$ is finitely generated for all $k \geq 2$, or
2. $\pi_k(N) \cong 0$ for $1 < k < n - 1$, in particular, aspherical manifold, or
3. $n \leq 4$.

### 2. Definitions and preliminaries

Throughout this paper, the symbols $\simeq$ and $\cong$ denote a homotopy equivalence and an isomorphism, respectively. All manifolds are understood to be finite dimensional, connected and metric.

For a closed manifold $N$, a proper map $p : M \to B$ is $N$-like if each fiber $p^{-1}(b)$ is shape equivalent to $N$. For simplicity, we shall assume that each fiber $p^{-1}(b)$ in an $N$-like map to be an $ANR$ having the homotopy type of $N$.

Let $N$ and $N'$ be (not necessarily closed) $n$-manifolds and $f : N \to N'$ be a map. Denote the $k$th cohomology group of $N$ with $G$-coefficients and compact supports by $H^k_c(N; G)$. If both $N$ and $N'$ are orientable, then the degree of $f$ is the nonnegative integer $d$ such that the induced endomorphism $f^* : H^k_c(N; \mathbb{Z}) \cong \mathbb{Z} \to H^k_c(N'; \mathbb{Z}) \cong \mathbb{Z}$ amounts to multiplication by $d$, up to sign. In general, the degree mod 2 of $f$ is the integer $d \in \{0, 1\}$ such that the induced endomorphism $f^* : H^k_c(N; \mathbb{Z}_2) \cong \mathbb{Z}_2 \to H^k_c(N'; \mathbb{Z}_2) \cong \mathbb{Z}_2$ amounts to multiplication by $d$.

Suppose that $N$ is a closed $n$-manifold and a proper map $p : M \to B$ is $N$-like. Let $G$ be the set of all fibers, that is, $G = \{p^{-1}(b) : b \in B\}$. Put $C = \{p(g) \in B : g \in G\}$ and there exist a neighbourhood $U_g$ of $g$ in $M$ and a retraction $R_g : U_g \to g$ such that $R_g | g' : g' \to g$ is a degree one map for all $g' \in G$ in $U_g$, and $C' = \{p(g) \in B : g \in G\}$ and there exist a neighbourhood $U_g$ of $g$ in $M$ and a retraction $R_g : U_g \to g$ such
that \( R_x | g' : g' \to g \) is a degree one mod 2 map for all \( g' \in G \) in \( U_x \). Call \( C \) the continuity set of \( p \) and \( C' \) the mod 2 continuity set of \( p \). Coram and Duvall showed [5] that \( C \) and \( C' \) are dense, open subsets of \( B \).

Call a closed manifold \( N \) Hopfian if it is orientable and every degree one map \( N \to N \) which induces a \( \pi_1 \)-isomorphism is a homotopy equivalence. A closed manifold \( N \) is \( s \)-Hopfian if \( N \) is Hopfian when \( N \) is orientable and \( N^H \) is Hopfian when \( N \) is non-orientable, where \( N^H \) is the covering space of \( N \) corresponding to \( H = \bigcap_{i \in I} \{ H_i : [\pi_1(N) : H_i] = 2 \} \). By Hall’s Theorem (for any finitely generated group \( G \), the number of subgroups of \( G \) having any fixed finite index is finite), the index set \( I \) is finite, and so \( H \) has a finite index in \( \pi_1(N) \). All closed manifolds with virtually nilpotent or finite fundamental group, all closed aspherical manifolds, and all closed \( n \)-manifolds (\( n \leq 4 \)) are examples of \( s \)-Hopfian manifolds. Whether all closed manifolds are \( s \)-Hopfian is related to the famous old problem of Hopf [14].

A group \( \Gamma \) is said to be Hopfian if every epimorphism \( f : \Gamma \to \Gamma \) is necessarily an isomorphism. A finitely presented group \( \Gamma \) is said to be hyper-Hopfian if every homomorphism \( f : \Gamma \to \Gamma \) with \( f(\Gamma) \) normal and \( \Gamma/f(\Gamma) \) cyclic is an isomorphism (onto). A group \( \Gamma \) is said to be residually finite if for any non-trivial element \( x \) of \( \Gamma \) there is a homomorphism \( f \) from \( \Gamma \) onto a finite group \( K \) such that \( f(x) \neq 1_K \). It is well known that every finitely generated residually finite group is Hopfian.

Given a group \( \Gamma \), we use \( \Gamma' \) to denote its commutator subgroup.

**Proposition 2.1 ([15] or [10]).** Let \( \psi : \Gamma \to \Gamma \) be an endomorphism of a finitely generated, residually finite group \( \Gamma \) with \( \Gamma' \subset \psi(\Gamma) \). Then there exists an integer \( k \geq 0 \) for which \( \psi \) restricts to a monomorphism on \( \psi^k(\Gamma) \). Moreover, if \( \Gamma / \Gamma' \) is finite, then \( \ker \psi \) is finite.

The next proposition is an easy consequence of the work of Epstein [13].

**Proposition 2.2 ([2, Lemma 3.2]).** Let \( M \) and \( N \) be manifolds and \( f : M \to N \) a proper map such that \( f_* : \pi_1(M) \to \pi_1(N) \) is an isomorphism. Let \( q' : N' \to N \) and \( q'' : M'' \to M \) be coverings such that \( q''(\pi_1(M'')) = f^{-1}(q_*(\pi_1(N'))) \). Suppose that \( f' : M'' \to N' \) is a lifting of \( f \circ q'' \) with \( f \circ q'' = f' \circ q' \). Then \( \deg f = \deg f' \in \mathbb{Z} \).

The following is basic for investigating co-dimension 2 fibrators.

**Lemma 2.3.** Let \( N \) be a closed \( s \)-Hopfian \( n \)-manifold with Hopfian fundamental group. If \( N \) is (F), then at least one of the following two cases occurs:

Case 1: There is an \( N \)-like proper map \( p : M^{n+2} \to \mathbb{R}^2 \) defined on an \( (n + 2) \)-manifold \( M \) which is an approximate fibration over \( \mathbb{R}^2 \setminus 0 \), but not an an approximate fibration over \( \mathbb{R}^2 \), such that \( p^{-1}(\mathbb{R}^2) \) is a strong deformation retract of \( p^{-1}(\mathbb{R}^2) \equiv (\text{say}) L \) under a retraction \( R : L \to p^{-1}(\mathbb{R}^2) \).
Case 2: There is an $N$-like proper map $p : M^{n+2} \to \mathbb{H}$ defined on an $(n+2)$-manifold $M$ which is an approximate fibration over $\mathbb{H}$, but not an approximate fibration over $\mathbb{H}$, such that $p^{-1}(0)$ is a strong deformation retract of $p^{-1}(\mathbb{H}) \equiv (\text{say}) L$ under a retraction $R : L \to p^{-1}(0)$ and for all $a \in \partial \mathbb{H}$, $R|p^{-1}(a) : p^{-1}(a) \to p^{-1}(0)$ is a homotopy equivalence, where $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, $\mathbb{H}_a = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and $\partial \mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$.

In either case, $(R|p^{-1}(x))_\#(\pi_1(p^{-1}(x))) \neq \pi_1(p^{-1}(0))$ for some $x(\neq 0) \in \mathbb{H}$.

**Proof.** If a closed s-Hopfian $n$-manifold $N$ with Hopfian fundamental group is (F), there is an $N$-like proper map $p : M^{n+2} \to B$ defined on an $(n+2)$-manifold $M$ which is not an approximate fibration. Hence $p : M^{n+2} \to B$ is not an approximate fibration at $x$ for some $x \in B$. Here $x \in C'$ or $x \in \partial B$.

For the case of $x \in C'$, applying [16, Theorem 3.1] and [7, Proposition 2.8], we can localize the situation into Case 1. Applying [7, Proposition 2.8] for the case of $x \in \partial B$, we can localize the situation into Case 2. In either case, the Hopfian hypotheses on $N$ and $\pi_1(N)$ gives $(R|p^{-1}(x))_\#(\pi_1(p^{-1}(x))) \neq \pi_1(p^{-1}(0))$ for some $x(\neq 0) \in \mathbb{H}$.

\[ \square \]

### 3. Proof of Main Theorem

Suppose that a closed s-Hopfian $n$-manifold $N$ with residually finite, torsion free $\pi_1(N)$ and finite $H_1(N)$ is (F). By Lemma 2.3, at least one of the two cases occurs. Since the method of the proof of Case 2 is basically same as Case 1, we only prove Case 1.

Put $g = p^{-1}(x)$ and $g_0 = p^{-1}(0)$. Take the covering $\alpha : L_1 \to L \equiv p^{-1}(\mathbb{H}^2)$ corresponding to $\text{incl}_\#(\pi_1(g))$. Take the covering $\beta : L_{1H} \to L_1$ corresponding to $H = \cap_i[H_i \leq \pi_1(L_1) : [\pi_1(L_1) : H_i] = 2]$ and then take the universal covering $\gamma : \tilde{L} \to L_{1H}$. Consider the following commutative diagram.

\[
\begin{array}{cccccc}
\tilde{g} & \xrightarrow{\text{incl}} & \tilde{L} & \xrightarrow{\tilde{\gamma}} & \tilde{g}_0 \\
| & | & | & | & | \\
\beta^{-1}(g) & \xrightarrow{\text{incl}} & L_{1H} & \xrightarrow{\beta} & g_{1H} \\
| & | & | & | & | \\
g & \xrightarrow{\text{incl}} & L_1 & \xrightarrow{\beta} & \alpha^{-1}(g_0) \\
| & | & | & | & | \\
g & \xrightarrow{\text{incl}} & L & \xrightarrow{\alpha} & g_0.
\end{array}
\]
Here, incl and $R_I$ are liftings of the inclusion map incl and $R \circ q$, respectively. $\tilde{g}$ and $\tilde{g}_0$ are the universal covering of $N$.

First, we claim that $R_I \circ incl$ induces a $\pi_1$-isomorphism.

Since $p$ is an approximate fibration over $\mathbb{R}^2 \setminus 0$, there is a homotopy exact sequence

$$\pi_1(g) \to \pi_1(p^{-1}(\mathbb{R}^2 \setminus 0)) \to \pi_1(\mathbb{R}^2 \setminus 0) \cong \mathbb{Z} \to 1$$

showing $\pi_1(p^{-1}(\mathbb{R}^2 \setminus 0))/\text{incl}(\pi_1(g)) \cong \mathbb{Z}$. Because $g$ has the homotopy type of a co-dimension 2 compactum from $L$, the inclusion $p^{-1}(\mathbb{R}^2 \setminus 0) \to L$ induces an epimorphism $\varphi$ of fundamental groups. It follows directly that $R_q \varphi \text{incl}(\pi_1(g))$ is a normal subgroup of $\pi_1(g_0)$ having cyclic cokernel. Hence, $\text{incl}_q(\pi_1(g))$ contains the commutator subgroup $\pi_1(L)$ of $\pi_1(L)$. Since $\pi_1(L) \cong \pi_1(g_0) \cong \pi_1(g) \cong \pi_1(N)$ is residually finite and $\pi_1(N)/\pi_1(N)' \cong H_1(N)$ is finite, by Proposition 2.1, $\ker(\text{incl}_q)$ is finite. But since $\pi_1(g)$ is torsion free, $\ker(\text{incl}_q)$ should be trivial, that is, $\text{incl}_q : \pi_1(g) \to \pi_1(L)$ is a monomorphism. Consequently, $(\text{incl}_q)_q : \pi_1(g) \to \pi_1(L)$ is an isomorphism, for $q_0 \circ (\text{incl}_q)_q = \text{incl}_q$ and $q_0$ is a monomorphism.

Since there is no upper semicontinuous decomposition of an orientable $(n + 2)$-manifold consisting entirely of nonorientable $n$-manifolds [7, Proposition 2.9], the orientability of $L_{gh}$ implies the orientability of $\beta^{-1}(g)$. So by [11, Lemma 5.5], the index $[\pi_1(g_0)] : (R_I \circ \text{incl})_q(\pi_1(\beta^{-1}(g)))$ equals to the degree of the map $R_I \circ incl$.

Applying the fact [18, Lemma 3.2] that $R_I \circ incl$ induces a $\pi_1$-isomorphism if and only if $R_I \circ \text{incl}$ induces a $\pi_1$-isomorphism, we see that the degree of the map $R_I \circ \text{incl}$ must be one.

(1) First assume that $\pi_k(N)$ is finitely generated for all $k \geq 2$.

By Proposition 2.2, we have that the degree of the map $R \circ incl$ is one. Hence $R \circ incl$ induces $H^*_c(X;G) \cong H_{\ast-k}(X;G)$ [21, page 388], $R \circ incl$ induces $H_{\ast-k}$-epimorphisms for all $k$. But since $\pi_k(N)$ is finitely generated for all $k \geq 2$, $H_{\ast-k}(\tilde{g}) \cong H_{\ast-k}(\tilde{g}_0)$ is a finitely generated Abelian group [22, page 509] (and so it is Hopfian) so that $(\tilde{R} \circ incl)_k : H_k(\tilde{g}) \to H_k(\tilde{g}_0)$ is an isomorphism. Appealing to the Hurewicz Theorem, we see that $\pi_k(\tilde{g}) \to \pi_k(\tilde{L})$ is an isomorphism for all $k \geq 2$. Whitehead’s Theorem ensures that the composition $g \to L_1 \to \alpha^{-1}(g_0)$ is a homotopy equivalence. But since $\alpha_l : \alpha^{-1}(g_0) \to g_0$ is a covering map, $g_0 \simeq N$ satisfies (CP).

(2) Next, assume that $\pi_k(N) \cong 0$ for $1 < k < n - 1$.

Recall the work of Swarup [23]: For a map $f : A \to B$ between closed oriented $n$-manifolds with $\pi_1$-isomorphism and $\pi_k(A) = \pi_k(B) = 0$ for $1 < k < n - 1$, $f$ is a homotopy equivalence if and only if $\deg f = 1$.

Since the degree of the map $R_I \circ incl$ is one, by the work of Swarup, $R_I \circ incl$ is a homotopy equivalence.

(3) Finally, assume that $n \leq 4$.

The case of $n = 3$ is a special case of (2).
For the case of \( n = 4 \), apply the following consequence of the work of Hausmann [14]: \textit{For any degree one map } \( f : A^4 \to B^4 \text{ between closed 4-manifolds with } \pi_1 \text{-isomorphism, } f \text{ is a homotopy equivalence.}\)

Although Hausmann only proves the case \( A = B \), just mimicking his proof and using the exact sequence of surgery with Poincaré duality, one may deduce the statement above.

### 4. Example and remarks

**Example ([12]).** A closed \( n \)-manifold \( N, n > 4 \), which fails to be a co-dimension 2 fibrator but \( H_1(N) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and every \( \gamma \in \pi_1(N), \gamma \neq 1 \), has infinite order.

Apply Maunder’s construction [20] to obtain a finite aspherical 2-complex \( K \) such that \( H_1(K) \cong \mathbb{Z}_2 \). Specify a PL embedding of \( K \) in an \( (n+1) \)-manifold \( M^{n+1} \), and let \( S \) be the boundary of a regular neighbourhood of the image. Let \( \Omega \) be the mapping cylinder of a 2-1 covering map \( \Theta : \overline{S} \to \overline{S} \); here \( \Omega \) is a (non-orientable) twisted \( I \)-bundle over \( S \). Form \( N \) by doubling \( \overline{\Omega} \) along \( \overline{S} \), its boundary.

A routine computation involving a Mayer-Vietoris sequence confirms \( H_1(N) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Note that \( \pi_1(\Omega) \cong \pi_1(S) \cong \pi_1(K) \), from which it follows that \( \pi_1(N) \cong \pi_1(\Omega) \ast_{\pi_1(\overline{S})} \pi_1(\Omega) \) is the fundamental group of an aspherical finite complex and, hence, no nontrivial element has finite order [1, Corollary VIII.2.5].

Such manifolds \( N \) fail to be co-dimension 2 fibrators, due to the existence of a 2-1 covering map \( \overline{N} \to N \) (see [7, Theorem 4.2]). For the most obvious 2-1 covering \( \overline{N} \to N \), \( \overline{N} \) will consist of two copies \( \Omega_1, \Omega_2 \) of \( \Omega \), arising as the preimage of one \( \Omega \) in the target space, \( N \), together with a 2-1 covering \( \overline{\Omega} \) of the other copy of \( \Omega \) used to form \( N \). But here \( \overline{\Omega} \) is simply \( \overline{S} \times [0,1] \), and \( \overline{N} = \Omega_1 \cup (\overline{S} \times [0,1]) \cup \Omega_2 \) with attachments that reveal \( \overline{N} \approx N \).

Note that for all \( i \geq 2, \pi_i(N \times S^k) (k \geq 2) \) is finitely generated.

**Remark.** The condition of torsion free \( \pi_1(N) \) cannot be omitted, since \( S^1 \times L(p,q) \) fails to be a co-dimension 2 fibrator ([9]) but it cannot cover itself cyclically, where \( L(p,q) \) is a Lens space.

On the other hand, the condition of finite \( H_1(N) \) is also imperative, since \( N \) fails to be a co-dimension 2 fibrator but it cannot cover itself cyclically, where \( N \) is some \( Nil \) 3-manifold (See [8]).

### References

Manifolds that fail to be co-dimension 2 fibrators

[5] ———, ‘Mappings from $S^1$ to $S^2$ whose point inverses have the shape of a circle’, General Topology Appl. 10 (1979), 239–246.

Department of Mathematics
Pusan National University
Pusan 609–735
Korea
e-mail: yhim@pusan.ac.kr

Department of Mathematics
Kyungpook National University
Taegu 702–701
Korea
e-mail: yongkuk@knu.ac.kr