ON THE NOTION OF RESIDUAL FINITENESS FOR $G$-SPACES

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Abstract

We define equivariant completion of a $G$-complex and define residually finite $G$-spaces. We show that the group of $G$-homotopy classes of $G$-homotopy self equivalences of a finite, residually finite $G$-complex, is residually finite. This generalizes some results of Roitberg.

Keywords and phrases: equivariant completion, residually finite $G$-spaces, residually finite groups, Hopfian groups.

1. Introduction

The notion of profinite completion in group theory is well understood and it is well known that profinite completion of a group is residually finite. The notion of profinite completion of Sullivan [8] in homotopy theory motivated Roitberg to introduce the notion of residual finiteness in the homotopy category [7]. He showed that the profinite completion of a path connected CW-complex is residually finite [7, Theorem 1 (a)]. He further showed that for a finite CW-complex $X$ which is residually finite, $\pi(X)$, the pointed homotopy classes of self homotopy equivalences is residually finite [7, Theorem 3]. This is the homotopy theoretic analogue of the well-known result of Baumslag that the automorphism group of a finitely generated residually finite group is residually finite. The aim of this paper is to prove equivariant versions of the above results of Roitberg.

Let $G$ be a finite group and $G-Hom$ denote the category of $G$-path connected $G$-CW-complexes (which we abbreviate to $G$-complexes) with base point. All maps and homotopies are based. Following Sullivan, we define the profinite completion
$\hat{X}_G$ of a $G$-complex $X$ (for equivariant completion, generalizing the non equivariant completion of Bousfield-Kan, see [3]). We also introduce the notion of residual finiteness for $G$-spaces and show that for any $X \in G\mathcal{H}$, the profinite completion $\hat{X}_G$ is residually finite. Let $\mathcal{E}_G(X)$ denote the group of $G$-homotopy classes of equivariant homotopy self equivalences of $X$. One of the main results of the paper is

**Theorem 1.1.** Let $X \in G\mathcal{H}$ be finite. Assume that $X$ is residually finite. Then $\mathcal{E}_G(X)$ is a residually finite group.

Recall that a theorem of Sullivan [9] and Wilkerson [11] says that if $X$ is a nilpotent finite complex, then $\mathcal{E}(X)$ is commensurable with an arithmetic group and hence, is finitely presented. Thus if $X$ is a finite, nilpotent complex which is also residually finite, then $\mathcal{E}(X)$ being residually finite and finitely presented, is Hopfian. The equivariant analogue of the Sullivan-Wilkerson theorem is proved in [10]. We use this to prove

**Theorem 1.2.** If $X \in G\mathcal{H}$ is finite and nilpotent, then $\mathcal{E}_G(X)$ is Hopfian.

**Convention** Throughout, $G$ will denote a finite group and all spaces, maps and homotopies are based and ‘$X \in G\mathcal{H}$ is finite’ is meant that $X$ is a finite $G$-CW-complex.

### 2. Equivariant completion and residual finiteness

Recall that a space $F$ is *totally finite* if the homotopy groups $\pi_n(F), n \geq 1$ are finite and if in addition there exists a positive integer $n_0$ such that $\pi_n(F) = 0$ for $n > n_0$. A space is of *finite type* if all its homotopy groups are finitely generated.

A $G$-space $X$ is *totally finite* if for every subgroup $H$ of $G$, the $H$ fixed point set $X^H$ is totally finite.

**Definition 2.1.** A $G$-space $X$ is *residually finite* if for any finite $G$-complex $W$ and $\alpha, \beta \in [W, X]_G, \alpha \neq \beta$ there exists a $G$-map $f : X \to Z$ with $Z$ totally finite such that $f_\ast(\alpha) \neq f_\ast(\beta)$ where $f_\ast : [W, X]_G \to [W, Z]_G$ is the map induced by $f$.

A $G$-map $f : X \to Y$ between $G$-spaces is a $\mathcal{T}$-*monomorphism* if for every finite $W \in G\mathcal{H}$ the induced map $f_\ast : [W, X]_G \to [W, Y]_G$ is a monomorphism.

Here is an example of a residually finite space.

**Example 2.2.** Let $X = S^1 \vee S^1$. Then $X$ can be given the structure of a $\mathbb{Z}_2$-complex as follows. $X$ has one 0-cell of the type $\mathbb{Z}_2/\mathbb{Z}_2$ and one 1-cell of the type $\mathbb{Z}_2/e$. $X$ can then be readily recognized as an equivariant Eilenberg-MacLane space $K(\lambda, 1)$.
where $\lambda$ is the $O_{Z_2}$-group $\lambda(Z_2/e) = F_2$, the free group of rank two, and $\lambda(Z_2/Z_2)$ is the trivial group. We claim that $X$ is residually finite as $Z_2$-space. First note that if $W$ is a finite $G$-complex then

$$[W, K(\lambda, 1)]_G \cong \text{Hom}_{O_G}(\pi_1(W), \lambda).$$

(This is true more generally [6]). Now let $\alpha, \beta \in [W, K(\lambda, 1)]_G$ be such that $\alpha \neq \beta$. Then clearly $\alpha(Z_2/e) \neq \beta(Z_2/e) : \pi_1(W) \rightarrow \lambda(Z_2/e)$. Since $F_2$ is residually finite there exists a finite group $F$ and a homomorphism $\mu : \lambda(Z_2/e) \rightarrow F$ such that $\mu \circ \alpha(Z_2/e) \neq \mu \circ \beta(Z_2/e)$. Define an $O_G$-group $\lambda'$ by $\lambda'(Z_2/e) = F$ and $\lambda'(Z_2/Z_2)$ to be the trivial group. Then, the map $\mu : \lambda \rightarrow \lambda'$ defined by $\mu(Z_2/e) = \mu$ and $\mu(Z_2/Z_2)$ being the trivial homomorphism, defines a natural transformation. This gives rise to a $G$-map $h : K(\lambda, 1) \rightarrow K(\lambda', 1)$ of equivariant Eilenberg-MacLane spaces. Clearly $h_*(\alpha) \neq h_*(\beta)$. Observe that $K(\lambda', 1)$ is totally finite. Note that $X$ is not nilpotent as a $Z_2$-space (compare Proposition 2.9).

**Proposition 2.3.** If $X$ is residually finite as a $G$-space, then $X^G$ is residually finite.

**Proof.** Let $\alpha, \beta \in [W, X^G]$, $\alpha \neq \beta$ with $W$ a finite CW-complex. Then endowing $W$ with the trivial $G$-action, $\alpha, \beta$ can be considered as elements of $[W, X]_G$ and it is easy to see that $\alpha \neq \beta$, as elements of $[W, X]_G$. Hence there is a totally finite $G$-space $Z$ and a $G$-map $f : X \rightarrow Z$, such that, $f_*(\alpha) \neq f_*(\beta)$. Then, it follows that $f^G_*(\alpha) \neq f^G_*(\beta)$. \qed

We can now construct a $G$-space $X$ which is residually finite, if one forgets the group action but is not residually finite when considered as a $G$-space.

**Example 2.4.** Let $G = Z_2$. Let $f : \mathbb{Q} \rightarrow \mathbb{Z}$ denote the only homomorphism between the additive group of rationals and the integers. This map is then realized as a map $f : K(\mathbb{Q}, 1) \rightarrow S^1$ of Eilenberg-MacLane spaces. Consider the $O_G$-space $T$, defined by, $T(G/G) = K(\mathbb{Q}, 1)$ and $T(G/e) = S^1$, with all structure maps as the identity, except the map $T(e) : T(G/G) \rightarrow T(G/e)$, which equals $f$. Then, by the Elmendorf construction [2], there exists a $G$-space $CT$, such that, $CT$ has the homotopy type of $S^1$, whereas $CT^G$ has the homotopy type of $K(\mathbb{Q}, 1)$. Corollary 1 of [7] shows that $CT^G$ is not residually finite, but the underlying space of the $G$-space $CT$, is clearly residually finite. It follows from the above proposition that, $CT$ is not residually finite, as a $G$-space.

We now turn to the definition of equivariant completion. Recall [4, Theorem 3.1, page 134] that, a contravariant functor from $G^\mathcal{C}$ to the category of sets, is representable, if and only if, it satisfies the Brown’s axioms (the wedge and the Mayer-Vietoris axioms). A functor satisfying the wedge and the Mayer-Vietoris axioms will
be called a **Brownian functor**. A **compact Brownian functor** is a Brownian functor taking values in compact Hausdorff spaces.

We shall need the following two properties of compact Brownian functors.

1. Suppose $k'$ is a contravariant functor defined on the subcategory of $\mathcal{M}$ consisting of finite $G$-complexes taking values in compact Hausdorff spaces. Suppose that $k'$ satisfies the Brown’s axioms, whenever they make sense. Then, there is a unique extension of $k'$ to a compact Brownian functor $k$, defined by, $k(X) = \text{inv lim}_a k'(X_a)$, where the inverse limit is over the finite $G$-subcomplexes $X_a$ of $X$.

2. The arbitrary inverse limit of compact Brownian functors, over a small filtering category, is a compact Brownian functor.

The proofs of both these facts are analogous to the nonequivariant case [8, page 36] and are therefore omitted. We shall use the above properties of compact Brownian functors to introduce equivariant completion as follows.

**Step 1** For $X \in \mathcal{M}$, let $\mathcal{F}_X$ denote the category whose objects are $G$-maps $X \to F$ with $F$ a totally finite $G$-space and morphisms are homotopy commutative diagrams.

**Lemma 2.5.** $\mathcal{F}_X$ is a small filtering category.

**Proof.** Recall ([8]) that, to show that the category $\mathcal{F}_X$ is small filtering we need to check the smallness, the directedness of $\mathcal{F}_X$ and the essential uniqueness of maps in $\mathcal{F}_X$. The first condition is clear since we can replace $\mathcal{F}_X$ by an equivalent small category, by picking a representative from each $G$-homotopy type of $F$’s. The second property is also clear as given objects $f_1 : X \to F_1$ and $f_2 : X \to F_2$ in $\mathcal{F}_X$ we can imbed them in $f_1 \times f_2 : X \to F_1 \times F_2$. The essential uniqueness of maps in $\mathcal{F}_X$ follows from the co-equalizer construction in equivariant homotopy theory, which is given by a suitable pushout diagram [4, page 39]. Explicitly, for two morphisms from $\pi' : X \to F'$ to $\pi : X \to F$ in $\mathcal{F}_X$ given by $G$-maps $f_1, f_2 : F' \to F$, consider the $G$-space

\[ \{(p, x) \in F^i \times F' : p(0) = f_1(x), p(1) = f_2(x)\} \]

with diagonal action, where the $G$-action on $F^i$ is induced by the action on $F$. Let $F''$ be the component of the above $G$-space containing the base point, the base point being the constant path at the base point of $F$ in the first factor and the base point of $F'$ in the second factor. Then, as in the non-equivariant case [4, page 40], we have an exact sequence

\[ \ldots \to \pi_i(F''^H) \to \pi_i(F'^H) \to \pi_i(F^H) \to \ldots , \]

for every subgroup $H$ of $G$. From this exact sequence it follows that $F''$ is a totally finite $G$-space. Now, one gets the required co-equalizer by using a $G$-homotopy from $f_1 \circ \pi'$ to $f_2 \circ \pi'$. □
Step 2 Let $Z \in G.\mathcal{X}$ be finite and $F$ a totally finite $G$-space. Then by equivariant obstruction theory [1], it is easy to see that, the homotopy set $[Z, F]_G$ is finite. This yields a contravariant functor defined on the sub category of $G.\mathcal{X}$ consisting of finite $G$-complexes and taking values in compact Hausdorff spaces. A direct verification shows that this functor satisfies the Brown’s axioms whenever they make sense. Then by property (1), we get a compact Brownian functor defined by $S_f(Y) = \liminv_{\alpha} [Y, F]_G = [Y, F]_G$, where the inverse limit is taken over the finite $G$-subcomplexes of $Y$.

From Step 1 and Step 2 we get a functor on $\mathcal{F}_X$ which assigns to each object $\pi : X \to F$, the compact Brownian functor $S_f$ obtained as in Step 2. By property (2) of compact Brownian functors $\liminv_{\alpha} S_f$ is again a compact Brownian functor, which assigns, to each $Y \in G.\mathcal{X}$, the compact Hausdorff space $\liminv_{\alpha} [Y, F]_G$. Therefore, by Brown’s representation theorem [4, Theorem 3.1, page 134], there exists a space $\hat{X}_G$ in $G.\mathcal{X}$ such that for every $G$-complex $Y$ there is a bijection

$$[Y, \hat{X}_G]_G \leftrightarrow \liminv_{\alpha} [Y, F]_G.$$

**Definition 2.6.** The space $\hat{X}_G$ is called the *equivariant profinite completion* of $X$.

Clearly, $\hat{X}_G$ comes equipped with a $G$-map $i : X \to \hat{X}_G$, which is determined by the objects of $\mathcal{F}_X$ and is called the completion map.

We now prove an important property of equivariant completion. First recall that a $G$-space $X$ is nilpotent if every fixed point set is nilpotent. An equivariant Postnikov decomposition for a $G$-space $B$ consists of $G$-maps $\alpha_n : B \to B_n$ and $r_{n+1} : B_{n+1} \to B_n$, $n \geq 0$ such that $B_0$ is a point and $\alpha_n$ induces an isomorphism $\pi q(\alpha_n) \to \pi q(B_n)$ for $q \leq n$, $r_{n+1} \alpha_{n+1} = \alpha_n$, and $r_{n+1}$ is the $G$-fibration over a $K(\pi q+1(B_n), n+2)$ by a map $k^{q+2} : B_n \to K(\pi q+1(B_n), n+2)$. On passage to $H$-fixed points, a Postnikov system for $B$ gives a Postnikov system for $B^H$. Moreover, every nilpotent $G$-space admits a Postnikov decomposition [4, 2].

**Proposition 2.7 (Hasse principle).** Let $Y \in G.\mathcal{X}$ be finite and $B \in G.\mathcal{X}$ be a nilpotent space of finite type. If $f, g : Y \to B$ are $G$-maps such that $i \circ f$ is $G$-homotopic to $i \circ g$, then $f$ is $G$-homotopic to $g$.

**Proof:** The proof is by induction over the stages in the equivariant Postnikov system of $B$ and is parallel to the nonequivariant case. Let $K \to B_{n+1} \to B_n$ be a part of the equivariant Postnikov decomposition of $B$ (see [4, 2]), where $K = K(\pi q, n+1)$ and $\pi q = \pi q+1(B_{n+1})$. Suppose $f_n : Y \to B_n$ and $f_{n+1} : Y \to B_{n+1}$ are the $G$-maps constructed from $f$. Now consider the $G$-fibration

$$\text{Map}(Y, K) \to \text{Map}(Y, B_{n+1}) \to \text{Map}(Y, B_n)$$
with the obvious action on the function spaces so that
\[ \Map(Y, B_{n+1})^G = \Map_G(Y, B_{n+1}). \]

We then have an ordinary fibration
\[ \Map_G(Y, K) \to \Map_G(Y, B_{n+1}) \to \Map_G(Y, B_n). \]

Consider the homotopy exact sequence of the above fibration
\[ \cdots \to \pi_1(\Map_G(Y, B_n), f_n) \to \pi_0(\Map_G(Y, K), f_{n+1}) \]
\[ f_{n+1}^* \to \pi_0(\Map_G(Y, B_{n+1}), f_{n+1}) \to \pi_0(\Map_G(Y, B_n), f_n). \]

Note that \( \pi_0(\Map_G(Y, K), f_{n+1}) = H_{n+1}^G(Y; \pi) \) where \( H_{n+1}^G(Y; \pi) \) denotes the Bredon cohomology group with coefficients in the \( O_G \)-group \( \pi \) [1]. Here \( f_{n+1}^* \) denotes the map given by the action of \( H_{n+1}^G(Y; \pi) \) on \( (f_{n+1})^* \in \pi_0(\Map_G(Y, K), f_{n+1}) \) obtained by equivariant obstruction theory [5]. Clearly, the image \( I = I(f_{n+1}) \) is the isotropy subgroup of the point \( (f_{n+1}) \) and the map \( r \) collapses the orbits of the action of \( H_{n+1}^G(Y; \pi) \). Thus we get an exact sequence
\[ 0 \to I(f_{n+1}) \to H_{n+1}^G(Y; \pi) \to \text{orbit}(f_{n+1}) \to 0. \]

We proceed as in the non-equivariant case and repeat the above argument for maps into completions \( \hat{\Map}_G \), to get a ladder whose top row being the above exact sequence, the base row being the exact sequence
\[ 0 \to I(\hat{f}_{n+1}) \to H_{n+1}^G(Y; \hat{\pi}) \to \text{orbit}(\hat{f}_{n+1}) \to 0, \]
and with induced maps \( c_0 : I(f_{n+1}) \to I(\hat{f}_{n+1}) \), \( c : H_{n+1}^G(Y; \pi) \to H_{n+1}^G(Y; \hat{\pi}) \) and \( c_1 : \text{orbit}(f_{n+1}) \to \text{orbit}(\hat{f}_{n+1}) \). Here, the \( O_G \)-group \( \hat{\pi} \) is defined by the group completion \( \hat{\pi}(G/H) = \pi(G/H) \). Also note that by property (1) of compact Brownian functor the map \( c : H_{n+1}^G(Y; \pi) \to H_{n+1}^G(Y; \hat{\pi}) \), is a finite completion. With this at our disposal the rest of the proof is exactly similar to the non-equivariant case. \( \square \)

Equivariant completion yields, as in the nonequivalent case ([7, Theorem 1]), examples of residually finite spaces.

**Proposition 2.8.** If \( X \in G\mathcal{H} \), then \( \hat{X}_G \) is residually finite.

Suppose that \( f : X \to Y \) is a \( G \)-map with \( Y \) residually finite. If \( f \) is a \( \mathbb{F} \)-monomorphism, then \( X \) is residually finite. The Hasse principle implies that if \( X \in G\mathcal{H} \) is nilpotent and of finite type, then the completion map \( i : X \to \hat{X}_G \) is a \( \mathbb{F} \)-monomorphism. Both these facts put together imply

**Proposition 2.9.** If \( X \in G\mathcal{H} \) is nilpotent and of finite type, then \( X \) is residually finite.
3. Proof of the main theorem

In this section we prove our main theorem which gives a sufficient condition for the group $\mathcal{E}_G(X)$ to be Hopfian. The main step in proving this (as in the non-equivariant case) is showing that, under suitable conditions, the group $\mathcal{E}_G(X)$ is residually finite.

**Definition 3.1.** Let $[f] : X \rightarrow Y$ be a morphism in $G\mathcal{H}$. $f$ is said to represent an epimorphism in $G\mathcal{H}$ if for any two maps $\alpha, \beta : Y \rightarrow Z$ in $G\mathcal{H}$, $\alpha \circ f$ is $G$-homotopic to $\beta \circ f$ implies $\alpha$ is $G$-homotopic to $\beta$.

Suppose that $X$ and $Y_0$ are in $G\mathcal{H}$ and $[X, Y_0]_G = \{[f_1], \ldots, [f_r]\}$. Define $Y = Y_0 \times \cdots \times Y_0$ with $r$ factors. Then $Y$ is a $G$-complex with the diagonal $G$ action. Consider the $G$-map $f : X \rightarrow Y$ by $f = (f_1, \ldots, f_r)$. Let $M(Y)$ denote the monoid of equivariant self homotopy equivalences of $Y$ preserving the base point. Each element of the symmetric group $S_r$ induces a self map of the $G$-space $Y$ by permuting its coordinates. This gives an embedding of $S_r$ into $M(Y)$.

**Lemma 3.2.** With the above notation, if $e : X \rightarrow X$ represents an epimorphism in $G\mathcal{H}$, then $e$ determines a unique $\sigma \in S_r \subseteq M(T)$ such that $f \circ e$ is $G$-homotopic to $\sigma \circ f$. The assignment $e \mapsto \sigma$ induces a monoid homomorphism $\psi : E(X) \rightarrow S_r \subseteq M(T)$, where $E(X)$ is the monoid of equivariant self epimorphisms of the $G$-space $X$.

**Proof of Theorem 1.1.** Let $\theta \in \mathcal{E}_G(X)$, $\theta \neq id$. We shall exhibit a homomorphism $\eta : \mathcal{E}_G(X) \rightarrow F$ with $F$ a finite group such that $\eta(\theta) \neq id$. Since $X$ is residually finite, we have a map $f : X \rightarrow Y_0$ of with $Y_0$ totally finite such that $f_*(\theta) \neq f_*(id)$. Since $X$ is finite and $Y_0^H$ is totally finite one observes using equivariant obstruction theory [1] that the equivariant homotopy set $[X, Y_0]_G$ is finite. Thus by Lemma 3.2 there is a $r > 1$ and a $\sigma \in S_r \subseteq M(Y_0)$ such that $f \circ \theta$ is $G$-homotopic to $\sigma \circ f$ and $f_*(\theta) \neq f_*(id)$. Hence $\sigma \neq 1$. Now the monoid homomorphism $\psi : E(X) \rightarrow S_r$ of Lemma 3.2 restricted to $M(X)$ induces a group homomorphism $\eta : \mathcal{E}_G(X) \rightarrow S_r$ such that $\eta(\theta) \neq id$. This completes the proof.

**Proof of Theorem 1.2.** Recall that by Proposition 2.9, $X$ is residually finite. Thus $\mathcal{E}_G(X)$ is a residually finite group. Moreover it follows from the work of Triantafillou [10, Theorem 1.2] that $\mathcal{E}_G(X)$ is commensurable with an arithmetic subgroup of $\mathcal{E}_G(X_0)$, where $X_0$ is the equivariant rationalisation of $X$. Thus $\mathcal{E}_G(X)$ is finitely generated. The theorem now follows as finitely generated residually finite groups are Hopfian. This completes the proof.

There are situations where it is not difficult to recognize the group $\mathcal{E}_G(X)$ as being residually finite.
EXAMPLE 3.3. Let $\lambda$ be an $O_G$-group. Let $n \geq 1$. If $n > 1$, then $\lambda$ is abelian. Then if $\lambda$ has the property that $\lambda(G/H)$ is finitely generated residually finite group for all subgroups $H$, then it is not difficult to see that $\partial_G(X)$ is residually finite where $X$ is the equivariant Eilenberg-MacLane space $K(\lambda, n)$.

EXAMPLE 3.4. As another example, suppose that $X \in G \mathcal{H}$ is a finite nilpotent space such that for any $G$-homotopy equivalence $f : X \to X$ which is not $G$-homotopic to identity, there exists a subgroup $H$ of $G$ such that $f^H : X^H \to X^H$ is not homotopic to the identity. Then $\partial_G(X)$ is residually finite (compare Proposition 3.5). We end with the following

PROPOSITION 3.5. Suppose $X \in G \mathcal{H}$ is a finite and nilpotent. Further assume that for each subgroup $H$, $K$ of $G$

1. $[X^K, X^H]$ is a group and
2. $[X^K, \Omega^n X^H]$ is trivial for $n \geq 1$.

Then $\partial_G(X)$ is residually finite.

PROOF. First note that for every subgroup $H$ of $G$, $X^H$ is nilpotent of finite type and hence $X^H$ is residually finite [7]. Now let $[f] \in \partial_G(X)$ such that $[f] \neq [id]$. Then there exists a subgroup $H$ of $G$ such that $[f^H] \neq [id]$, otherwise, by [2, Theorem 3], the natural family $\{[f^H]\}$ would correspond to $id : X \to X$ and this would mean $f \simeq_G id$. The group $\partial'(X^H)$ is residually finite by [7, Theorem 3]. Using the obvious homomorphism $\partial_G(X) \to \partial'(X^H)$ one sees that the group $\partial_G(X)$ is also residually finite. This completes the proof. 

COROLLARY 3.6. Suppose $X \in G \mathcal{H}$ is a finite and nilpotent. Moreover suppose that the $G$-action on $X$ is free outside the base point. Then $\partial_G(X)$ is residually finite.

EXAMPLE 3.7. Let $X = S^2 \vee S^2$. Then $X$ can be given a $\mathbb{Z}_2$-complex structure by interchanging the copies of $S^2$. Then $X$ satisfies the hypothesis of the corollary and hence $\partial_G(X)$ is residually finite. It is easy to see that this group is non-zero.

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References


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