VECTOR VALUED MEAN-PERIODIC FUNCTIONS ON GROUPS

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Abstract

Let $G$ be a locally compact Hausdorff abelian group and $X$ be a complex Banach space. Let $C_c(G; X)$ denote the space of all continuous functions $f : G \to X$, with the topology of uniform convergence on compact sets. Let $X'$ denote the dual of $X$ with the weak* topology. Let $M_c(G; X')$ denote the space of all $X'$-valued compactly supported regular measures of finite variation on $G$. For a function $f \in C_c(G; X)$ and $\mu \in M_c(G; X')$, we define the notion of convolution $f \ast \mu$. A function $f \in C_c(G; X)$ is called mean-periodic if there exists a non-trivial measure $\mu \in M_c(G; X')$ such that $f \ast \mu = 0$. For $\mu \in M_c(G; X')$, let $MP(\mu) = \{f \in C_c(G; X) : f \ast \mu = 0\}$ and let $MP(G, X) = \bigcup \mu \in M_c(G; X') M\mu$. In this paper we analyse the following questions: Is $MP(G, X) \neq \emptyset$? Is $MP(G, X) \neq C(G, X)$? Is $MP(G, X)$ dense in $C(G, X)$? Is $MP(\mu)$ generated by ‘exponential monomials’ in it? We answer these questions for the groups $G = \mathbb{R}$, the real line, and $G = \mathbb{T}$, the circle group. Problems of spectral analysis and spectral synthesis for $C(\mathbb{R}, X)$ and $C(\mathbb{T}, X)$ are also analysed.

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1. Introduction

The notion of mean-periodic functions was introduced in 1935 by Delsarte [5]. It is well known that every solution of a constant coefficient homogeneous ordinary differential equation is a finite linear combination of solutions of the type $t^k e^{\lambda t}$, where $\lambda \in \mathbb{C}$, and $k \in \mathbb{Z}_+$. Delsarte was interested in knowing whether this result is still true for convolution equation of the following type

$$\int_{\mathbb{R}} f(s - t)k(t) \, dt = 0, \quad \forall s \in \mathbb{R},$$

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where $k$ is a continuous function which is zero outside some interval. For $\tau > 0$, periodic continuous functions of period $\tau$ are solutions of the convolution equation

$$\frac{1}{\tau} \int_{t-t/\tau}^{t+t/\tau} f(t) \, dt = 0, \quad \forall s \in \mathbb{R}. \tag{2}$$

For this reason Delsarte called the continuous functions which are solutions of equation (1) as *mean-periodic*. In [35], Schwartz observed that the mean-periodicity of a continuous function does not depend upon the function $k$, and he extended Delsarte’s definition as follows:

**Definition 1.1.** A continuous function $f : \mathbb{R} \to \mathbb{C}$ is said to be *mean-periodic* if there exists a non-trivial regular measure $\mu$ of compact support and finite variation such that $(f \ast \mu)(s) = \int_{\mathbb{R}} f(s-t) \, d\mu(t) = 0, \forall s \in \mathbb{R}.$

Schwartz also gave an intrinsic characterization of mean-periodic functions. Let $C(\mathbb{R})$ denote the vector space of complex valued continuous functions on $\mathbb{R}$ with the topology of uniform convergence on compact sets (u.c.c.). Let $M_c(\mathbb{R})$ denote the space of all regular measures of compact support and finite variation on $\mathbb{R}$. For $f \in C(\mathbb{R})$, let $\tau(f)$ denote the closed translation invariant subspace of $C(\mathbb{R})$ generated by $f$. Schwartz in [35] showed that $f \in C(\mathbb{R})$ is mean-periodic if and only if $\tau(f) \neq C(\mathbb{R})$.

Further, if $f \ast \mu = 0$ for some non-zero $\mu \in M_c(\mathbb{R})$, then $f$ is a limit of finite linear combination of exponential monomials $t^k e^{it}$ which satisfy $t^k e^{it} \ast \mu = 0$. More generally, convolution equation of the type

$$f \ast \mu = g, \tag{3}$$

where $\mu \in M_c(\mathbb{R})$ and $g \in C(\mathbb{R})$ are given, can be analysed as in the case of ordinary differential equations. If $p$ is a particular solution of the equation (3), then every other solution is of the form $h + p$, where $h$ is a solution of the homogeneous equation $f \ast \mu = 0$. In general, equation (3) need not have any solution in $C(\mathbb{R})$. For instance, let $\mu$ be such that $d\mu(t) = \phi(t) \, dt$, where $\phi \in C^\infty(\mathbb{R})$, space of all infinitely differentiable functions on $\mathbb{R}$, and $g$ is a nowhere differentiable continuous function on $\mathbb{R}$. Some particular cases of (3) were analysed in [31, 32]. In general, no necessary and sufficient conditions for the existence of solutions of equation (3) are known. A variant of the above problem is the following: Consider the following convolution equation

$$f_1 \ast \mu_1 = -f_2 \ast \mu_2, \tag{4}$$

where $\mu_1, \mu_2 \in M_c(\mathbb{R})$ are given. Equation (4) can be written as a convolution equation for vector valued functions: let $\underline{f} = (f_1, f_2) : \mathbb{R} \to \mathbb{C}^2$ and $\underline{\mu} = (\mu_1, \mu_2) : \mathbb{R} \to \mathbb{C}^2$. 
Then equation (4) is a homogeneous equation $f \ast \mu = 0$. This leads to consideration of vector valued mean-periodic functions, the main content of this paper. We consider such equations in a more general setting and analyse their solutions.

Let $G$ be a locally compact abelian group. Let $X$ be a complex Banach space and $X'$ denote the weak* dual of $X$. We denote by $\mathcal{B}_G$ the $\sigma$-algebra of Borel subsets of $G$. We recall some results on integration of functions $f : G \to X$ with respect to $X'$-valued measures on $\mathcal{B}_G$, denoted by $M(G, X')$. For details one may refer Schmets [34]. Let $\mu \in M(G, X')$ and for every $x$, let $\mu_x$ denote the scalar measure on $\mathcal{B}_G$ defined by $\mu_x(E) := \langle x, \mu(E) \rangle$ for every $E \in \mathcal{B}_G$. The measure $\mu$ is said to be regular if $\mu_x$ is regular for every $x \in X$. For $E \in \mathcal{B}_G$, if $E = \bigcup_{i=1}^n E_i$ for some $E_1, E_2, \ldots, E_n \in \mathcal{B}_G$ such that $E_i \cap E_j = \emptyset$ for $i \neq j$, we call $\{E_1, E_2, \ldots, E_n\}$ a measurable partition of $E$. Let $\mathcal{P}(E)$ denote the set of all measurable partitions of $E$. Let

$$V_\mu(E) := \sup \left\{ \sum_{i=1}^n \||\mu_\mu(E_i)\| : \{E_1, E_2, \ldots, E_n\} \in \mathcal{P}(E) \right\}.$$ 

The scalar measure $V_\mu$ is called the variation of $\mu$. We say $\mu$ has finite variation if $V_\mu(E) < +\infty$ for every $E \in \mathcal{B}_G$. Let $M(G, X')$ denote the set of all regular Borel measures $\mu$ on $G$ such that $\mu$ has finite variation. For $\mu \in M(G, X')$ the smallest closed set $S$ with $\mu(E) = 0$ for every $E \in \mathcal{B}_G$ with $E \cap S = \emptyset$ is called the support of $\mu$. We write $S = \text{supp}(\mu)$ if $S$ is the support of $\mu$. Let $M_1(G, X')$ denote the set of all $\mu \in M(G, X')$ such that support of $\mu$ is compact. Let $C(G, X)$ denote the space of all $X$-valued continuous functions on $G$ with the topology of uniform convergence on compact sets. Let $f \in C(G, X)$ and $\mu \in M_1(G, X')$ with $\text{supp}(\mu) \subseteq K$, a compact set. Then there exists a sequence $\mathcal{P}_k(K) := \{B_{k,1}^i, B_{k,2}^i, \ldots, B_{k,n}^i\}$ of measurable partitions of $K$ with the following property: for arbitrary choice of $t_i \in B_{k,i}$, the sequence $\{\sum_{i=1}^n \langle f(t_i), \mu(B_{k,i}) \rangle\}_{k \in \mathbb{Z}^+}$ is convergent and is independent of the choice of $t_i$s. This limit is called the integral of $f$ with respect to $\mu$ and is denoted by $\int f \, d\mu$. For $f \in C(G, X)$ and $\mu \in M_1(G, X')$ the scalar valued function

$$(f \ast \mu)(g) := \int_G f(g - h) \, d\mu(h), \quad \forall \ g \in G$$

is called the convolution of $f$ with $\mu$, that is, $(f \ast \mu)(g) = \mu(f_g) = \langle \mu, f_g \rangle$, where $f_g(h) = f(g + h)$ and $(\mu, f) = \mu(f) = \int_G f(-g) \, d\mu(g)$ is the duality pairing of $M_1(G, X')$ with $C(G, X)$.

**Definition 1.2.** We say $f \in C(G, X)$ is mean-periodic if there exists a non-trivial $\mu \in M_1(G, X')$ such that $(f \ast \mu)(g) = \int_G f(g - h) \, d\mu(h) = 0, \forall \ g \in G$.

The aim of this paper is to answer the following questions: let $MP(G, X)$ denote the space of all $X$-valued mean-periodic functions on $G$. 

Is\( MP(G, X) \neq \emptyset \)? That is, when does there exist non-zero mean-periodic functions?

Is\( MP(G, X) \neq C(G, X) \)? That is, do there exist continuous functions which are not mean-periodic?

Is\( MP(G, X) \) dense in \( C(G, X) \)? That is, how large is \( MP(G, X) \) as a subspace of \( C(G, X) \)?

We answer these questions for the particular cases \( G = \mathbb{R} \), in Section 2 and \( G = \mathbb{T} \), circle group, in Section 3. Analysis of such questions for more general groups remain open.

The problem of analysing mean-periodic functions is also related to the problem of 'spectral analysis' and 'spectral synthesis'. In order to carry-out the analysis, we define next vector valued exponential monomials and exponential polynomials.

An additive function on a locally compact abelian group is a complex valued continuous function \( a \) on \( G \) such that \( a(g_1 + g_2) = a(g_1) + a(g_2) \) for all \( g_1 \) and \( g_2 \) in \( G \). A polynomial on \( G \) is a function of the form \( p(a_1, a_2, \ldots, a_m) \), where \( p \) is a polynomial in \( m \) variables and \( a_1, a_2, \ldots, a_m \) are additive functions on \( G \). A monomial on \( G \) is a function of the form \( p(a_1, a_2, \ldots, a_m) \), where \( p \) is a monomial in \( m \) variables and \( a_1, a_2, \ldots, a_m \) are additive functions on \( G \). An exponential on \( G \) is a non-zero continuous complex valued function \( \omega \) such that \( \omega(g_1 + g_2) = \omega(g_1)\omega(g_2) \) for all \( g_1 \) and \( g_2 \) in \( G \). An exponential monomial is a point-wise product of a monomial and an exponential. An exponential polynomial is a point-wise product of a polynomial and an exponential. The set of all exponentials is denoted by \( \Omega \). Note that \( \Omega \subset C(G) \).

We define exponential polynomials in \( C(G, X) \) as follows:

**Definition 1.3.**

(i) We call \( f \in C(G, X) \) an \( X \)-valued exponential if for every \( g \in G \), \( f(g) = \omega(g)x \) for some \( \omega \in \Omega \) and \( x \in X \).

(ii) We call \( f \in C(G, X) \) an \( X \)-valued exponential monomial if for every \( g \in G \), \( f(g) = p(g)\omega(g)x \) for some \( x \in X \), \( p \) a monomial in \( C(G) \) and \( \omega \) an exponential in \( C(G) \).

(iii) We call \( f \in C(G, X) \) an \( X \)-valued exponential polynomial if for every \( g \in G \), \( f(g) = p(g)\omega(g)x \) for some \( x \in X \), \( p \) a polynomial in \( C(G) \) and \( \omega \) an exponential in \( C(G) \).

**Example 1.**

(1) Let \( f \in C(\mathbb{R}, X) \). Then \( f \) is an exponential if and only if for every \( t \in \mathbb{R} \), \( f(t) = e^{\lambda t}x \) for some \( \lambda \in \mathbb{C} \) and \( x \in X \). \( f \) is an exponential monomial if and only if for every \( t \in \mathbb{R} \), \( f(t) = t^k e^{\lambda t}x \) for some \( \lambda \in \mathbb{C} \), \( k \in \mathbb{N} \) and \( x \in X \). Finally, \( f \) is an exponential polynomial if and only if for every \( t \in \mathbb{R} \), \( f(t) = p(t)e^{\lambda t}x \) for some \( \lambda \in \mathbb{C} \), polynomial \( p(t) \) and \( x \in X \). Thus the exponentials, exponential monomials and exponential polynomials are the scalar multiples of the ones defined by Schwartz [35].
A function \( f \in C(\mathbb{T}, X) \) is an exponential if and only if for every \( t \in \mathbb{R} \), \( f(e^{it}) = e^{int}x \) for some non-negative integer \( n \) and \( x \in X \).

**Remark.** We shall use the following convention: When \( X = \mathbb{C} \) we choose the \( x \in X \) appearing in the exponential, exponential monomial and exponential polynomial to be the scalar constant 1. The generality is not lost due to this choice, since if a closed translation invariant subspace contains an exponential or exponential monomial or exponential polynomial if and only if it contains their scalar multiples.

**Definition 1.4.** Let \( V \) be a closed translation invariant subspace of \( C(G, X) \). We say

(i) **spectral analysis holds for** \( V \) if \( V \) contains an exponential;

(ii) **spectral synthesis holds for** \( V \) if the linear span of the set of all exponential monomials in \( V \) is dense in \( V \);

(iii) if spectral analysis (synthesis) holds for every closed translation invariant subspace \( V \) of \( C(G, X) \), then we say that **spectral analysis (synthesis) holds in** \( C(G, X) \).

**Definition 1.5.** Let \( V \) be a closed translation invariant subspace of \( C(G, X) \) and \( f \in C(G, X) \) be mean-periodic. Let \( \tau(f) \) denote the closed translation invariant subspace of \( C(G, X) \) generated by \( f \).

(i) The **spectrum** of \( V \) is defined to be the set of all exponential monomials in \( V \) and is denoted by \( \text{spec}(V) \) or \( \sigma(V) \).

(ii) The **spectrum** of \( f \) is defined to be \( \text{spec}(\tau(f)) \) and is denoted by \( \text{spec}(f) \) or \( \sigma(f) \).

Some of the known results for spectral analysis and spectral synthesis for \( G = \mathbb{R}^n \) are as follows: Let \( E(\mathbb{R}^n) \) be the space of all infinitely differentiable functions on \( \mathbb{R}^n \) in the topology of compact convergence of functions and their derivatives. Then its dual \( E(\mathbb{R}^n)' \) is the space of all compactly supported distributions on \( \mathbb{R}^n \). Schwartz [35] proved the following theorem:

**Theorem 1.6 ([35]).** In \( E(\mathbb{R}) \), every closed translation invariant subspace is the closure of finite linear combinations of the exponential monomials in it.

As a consequence of this theorem, the linear span of exponential monomials in every closed translation invariant subspace \( V \) of \( C(\mathbb{R}) \) is dense in \( V \). That is, spectral analysis and spectral synthesis hold in \( C(\mathbb{R}) \). Using this Schwartz [35] described mean-periodic functions on \( \mathbb{R} \).

Let \( V \) be the closed translation invariant subspace of \( E(\mathbb{R}^n) \) generated by the solutions of the homogeneous constant coefficient partial differential equation \( p(D)f = 0 \). Malgrange [28] proved that spectral synthesis holds for \( V \).
In 1975 Gurevich [17] proved that Theorem 1.6 cannot be extended for $\mathbb{R}^n$, $n > 1$. Though Theorem 1.6 fails for $\mathbb{R}^n$, $n > 1$, spectral analysis and spectral synthesis hold in $C(G)$ for certain groups, for example, for $G = \mathbb{Z}^n$ (see [26]) and for discrete abelian groups (see [12, 13]). Consider the following example from [15].

**Example 2 ([10, 15]).** Define $f_1, f_2 : \mathbb{R}^2 \to \mathbb{C}$ by

$$f_1(x_1, x_2) := 1 \quad \text{and} \quad f_2(x_1, x_2) := x_1 + x_2, \quad \forall (x_1, x_2) \in \mathbb{R}^2.$$ 

Let $V$ be the closed translation invariant subspace of $C(\mathbb{R}^2)$ generated by $f_1$ and $f_2$. Then the spectrum of $V$ is $\{f_1\}$. But the closed linear span of the spectrum of $V$ is a proper subspace of $V$. Thus spectral synthesis fails in $C(\mathbb{R}^2)$ and spectral synthesis fails for $V$ even if $V$ is finite dimensional.

However, for certain closed translation invariant subspaces $V \subset C(\mathbb{R}^2)$ the linear span of all exponential polynomials in $V$ is dense in $V$. These subspaces are described in the following three theorems.

**Theorem 1.7 ([4]).** Let $V$ be a closed translation and rotation invariant subspace of $C(\mathbb{R}^2)$. Then the linear span of exponential polynomials in $V$ is dense in $V$.

**Theorem 1.8 ([16]).** Let $\mu \in M_\mu(\mathbb{R}^2)$. Then the linear span of exponential polynomials in $\tau_\mu := \{f \in C(\mathbb{R}^2) : f * \mu = 0\}$ is dense in $\tau_\mu$.

**Theorem 1.9 ([14]).** Let $V$ be a finite dimensional translation invariant subspace of $C(\mathbb{R}^3)$. Then every element of $V$ is a finite linear combination of exponential polynomials.

The following question is raised in [15] and the answer is not known: Let $V$ be closed translation invariant subspace of $C(\mathbb{R}^5)$.

- Does there exist an exponential in $V$?

In Section 4, we answer this question affirmatively when $V$ is either finite dimensional or rotation invariant or $V = \tau_\mu := \{f \in C(\mathbb{R}^2) : f * \mu = 0\}$ for some $\mu \in M_\mu(\mathbb{R}^2)$.

Let $V$ be a closed translation invariant subspace of $C(G, X)$. Then the problems of spectral analysis and synthesis are the following:

- Is every exponential monomial in $C(G, X)$ mean-periodic?
- Are exponential monomials dense in $C(G, X)$?
- When does there exist an exponential monomial in $V$?
- When is the linear span of exponential monomials in $V$ dense $V$?
- Does there exist an exponential monomial solution for the convolution equation $f * \mu = 0$ for a given $\mu \in M_\mu(G, X')$?

We analyse these problems for $G = \mathbb{R}$ in Section 2 and $G = \mathbb{T}$ in Section 3.
2. Mean-periodic functions on $G = \mathbb{R}$

For $G = \mathbb{R}$ and $X = \mathbb{C}$, it is known (see Schwartz [35]) that $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic if and only if $\tau(f)$, the closed translation invariant subspace of $C(\mathbb{R}, \mathbb{C})$ is proper. We first extend this result to $X$, arbitrary Banach space.

**Theorem 2.1.** The following are equivalent:

(i) $f$ is mean-periodic;

(ii) $\tau(f) \neq C(\mathbb{R}, X)$.

**Proof.** We use the fact that $C(\mathbb{R}, X)$ is a locally convex space and its dual is $M^*_c(\mathbb{R}, X')$. To show that (i) implies (ii): let $\mu \in M^*_c(\mathbb{R}, X')$ be non-trivial such that $f \ast \mu = 0$. Then $\mu(g) = 0$ for every $g \in \tau(f)$. Hence $\tau(f) \neq C(\mathbb{R}, X)$, for otherwise $\mu(g) = 0$ for every $g \in C(\mathbb{R}, X)$, which is not possible, since $\mu$ is non-trivial. The implication (ii) implies (i) follows from the Hahn-Banach theorem for locally convex spaces and the fact that $\tau(f)$ is a proper closed translation invariant subspace of $C(\mathbb{R}, X)$. □

We show next that there exist nontrivial $X$-valued mean-periodic functions on $\mathbb{R}$.

**Proposition 2.2.** $MP(\mathbb{R}, X) \neq \emptyset$.

**Proof.** Let $0 \neq x \in X$ and $0 \neq x' \in X'$. Choose $g \in MP(\mathbb{R})$, scalar valued function mean-periodic with respect to some $\mu \in M^*_c(\mathbb{R})$. Define $\nu : \mathbb{B}_{\mathbb{R}} \to X'$ by $\nu(E) := \mu(E)x'$ and define $f : \mathbb{R} \to \mathbb{C}$ by $f(t) := g(t)x$. Then $\mu$ is a $X'$-valued measure and $f$ is a continuous $X$-valued function with $f \ast \nu = (g \ast \mu)(x, x') = 0$. Thus $f$ is mean-periodic with respect to $\nu$. □

We prove next that existence of functions which are not mean-periodic is related to the $X$ being separable.

**Theorem 2.3.** $MP(\mathbb{R}, X)$ is a proper subset of $C(\mathbb{R}, X)$ if and only if $X$ is separable.

**Proof.** Suppose that $X$ is a non-separable complex Banach space and $f \in C(\mathbb{R}, X)$. Since $f$ continuous, $f(\mathbb{R})$ is separable and hence $\overline{f(\mathbb{R})}$ is separable. Since, for every $g \in \tau(f)$, $g(\mathbb{R}) \subseteq \overline{f(\mathbb{R})}$, $\tau(f) \neq C(\mathbb{R}, X)$. Hence $f$ is mean-periodic.

Conversely, suppose that $X$ is separable. We show that $MP(\mathbb{R}, X) \neq C(\mathbb{R}, X)$. For every $n \in \mathbb{N}$, let

$$f_n(t) := \sum_{j=1}^{\infty} a_{nj} e^{ij \omega t}, \quad t \in \mathbb{R},$$

(5)
where \( \lambda_{nj} \) and \( a_{nj} \) satisfy the following conditions:

(i) \( 0 \neq a_{nj} \in \mathbb{C} \).

(ii) \( \lambda_{nj} \in [\alpha, \beta] \) for some \( \alpha < \beta \).

(iii) \( \{\lambda_{nj} : j \in \mathbb{N}\} \cap \{\lambda_{nm} : j \in \mathbb{N}\} = \emptyset \) for \( m \neq n \) and for every \( n \), \( \{\lambda_{nj}\}_{j=1}^{\infty} \) has a limit \( \lambda_n \in \mathbb{R} \).

(iv) The convergence in (5) is uniform on compact sets with each \( f_n \) bounded by 1.

(v) \( \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |a_{nj}| < \infty \).

Let \( \{x_1, x_2, \ldots\} \) be a dense subset of \( X \). Define \( f : \mathbb{R} \to X \) by

\[
(6) \quad f(t) := \sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} f_n(t) x_n, \quad t \in \mathbb{R}.
\]

We show that \( f \) is not mean-periodic. Since \( \{e^{i\lambda_{nj}t}\}_{n,j=1}^{\infty} \) is an equicontinuous family, \( \{f_n\}_{n=1}^{\infty} \) is an equicontinuous family. Therefore, for \( \mu \in M_1(\mathbb{R}, X') \),

\[
\sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} (f_n \ast \mu)(t) = 0, \quad \forall t \in \mathbb{R},
\]

that is, for every \( t \in \mathbb{R} \),

\[
\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} a_{nj} \hat{\mu}_{x_n}(\lambda_{nj}) e^{i\lambda_{nj}t} = 0.
\]

Let \( S_{pq}(t) = \sum_{n=1}^{p} \sum_{j=1}^{q} e^{i\lambda_{nj}t} a_{nj} \hat{\mu}_{x_n}(\lambda_{nj}) / 2^n(1 + \|x_n\|) \). Notice that \( S_{pq} \) is almost periodic and its Fourier coefficients \( a(S_{pq}; \lambda) \) satisfy the following:

\[
\text{(8) } a(S_{pq}; \lambda) = \begin{cases} 
\frac{a_{nj} \hat{\mu}_{x_n}(\lambda_{nj})}{2^n(1 + \|x_n\|)} & \text{if } \lambda = \lambda_{nj}, \ 1 \leq n \leq p, \ 1 \leq j \leq q; \\
0 & \text{otherwise}.
\end{cases}
\]

Since the convergence in (6) is uniform, the convergence in (7) also is uniform. Therefore \( S_{pq} \) converges to 0 uniformly as \( p, q \to \infty \). Further, the Fourier coefficients \( a(S_{pq}; \lambda) \) converges to 0 as \( p, q \to \infty \) ([27]). In view of (8), \( a(S_{pq}; \lambda) = 0 \) for every \( \lambda \). Moreover, \( \hat{\mu}_{x_n}(\lambda_{nj}) = 0 \) for every \( n \) and \( j \). Since \( \{\lambda_{nj}\}_{j=1}^{\infty} \) has limit point, this implies \( \mu_{x_n} = 0 \) for all \( n \). Therefore, \( \mu = 0 \). Hence \( f \) is not mean-periodic.

Let \( f \in C(\mathbb{R}, X) \) and let \( x' \in X' \). Then \( x' \circ f \in C(\mathbb{R}) \). It is natural to ask the following question: Is \( x' \circ f \) mean-periodic for every \( x' \neq 0 \) if \( f \) is mean-periodic? We analyse this in the following theorem.
THEOREM 2.4. For \( f \in C(\mathbb{R}, X) \) and \( x', y' \in X' \) with \( x' \neq y' \) the following hold:

(i) If \( x' \circ f \) is mean-periodic, then \( f \) is mean-periodic.
(ii) If \( x' \circ f = y' \circ f \), then \( f \) is mean-periodic.
(iii) If \( X = \mathbb{C}^n, n > 1 \), then \( f \) is a finite sum of mean-periodic functions.
(iv) There exists \( f \in MP(\mathbb{R}, \mathbb{C}^n) \) such that \( x' \circ f \) is not mean-periodic for any \( x' \in X', x' \neq 0 \).

PROOF. (i) By Theorem 2.1, it suffices to show that \( \tau(f) \neq C(\mathbb{R}) \). For this, let \( g \in C(\mathbb{R}) \), \( g \neq 0 \) be such that \( g \notin \tau(x' \circ f) \). Choose \( v \in X \) such that \( \langle x', v \rangle \neq 0 \) and define \( h : \mathbb{R} \to X \) by \( h(t) = g(t)v / \langle x', v \rangle \). Then \( h \) is continuous and \( \langle x' \circ h(t) \rangle(t) = g(t) \). We show that \( h \) is not in \( \tau(f) \). If possible let, \( h \in \tau(f) \). Then there exists \( \sum c_if_i \to h \), which implies \( x' \sum c_if_i \to x' \circ h = g \), a contradiction.

(ii) Choose \( g \in C(\mathbb{R}, X) \) such that \( x'(g) \neq y'(g) \). We show that \( g \notin \tau(f) \). If possible let, \( g \in \tau(f) \). Since \( \sum c_if_i \to g \Rightarrow x'(\sum c_if_i) \to x'(g) \) and \( y'(\sum c_if_i) \to y'(g) \), and also since \( x'(f) = y'(f) \), \( x'(\sum c_if_i) = y'(\sum c_if_i) \). This implies \( x'(g) = y'(g) \), a contradiction.

(iii) Let \( f = (f_1, f_2, \ldots, f_n) \). Obviously \( (0, \ldots, 0, f_i, 0, \ldots, 0) \) is mean-periodic for every \( i \) with respect to \( \mu = (\mu_1, \ldots, \mu_n) \) where \( 0 \neq \mu_j \in M_c(\mathbb{R}) \) are arbitrary and for \( j = i, \mu_j = 0 \). Hence \( f \) is a finite sum of mean-periodic functions.

(iv) Choose a non zero, compactly supported complex valued continuous function \( g \). Let \( f = (g, g, \ldots, g) \). Then \( f \) is a \( \mathbb{C}^n \)-valued continuous function on \( \mathbb{R} \). Clearly \( f \) is mean-periodic with respect to \( \mu = (\nu_1, -\nu_1, 0, \ldots, 0) \), where \( 0 \neq \nu_1 \in M_c(\mathbb{R}) \) is arbitrary but \( x' \circ f \) is not mean-periodic for any \( 0 \neq x' \in X' \).

REMARK. When \( X = \mathbb{C}, MP(\mathbb{R}, X) \) is a subspace of \( C(\mathbb{R}, X) \). It follows from Theorem 2.4 (iii) that sum of mean-periodic functions in \( C(\mathbb{R}, X) \) need not be mean-periodic and hence \( MP(\mathbb{R}, X) \) in general need not be a vector subspace of \( C(\mathbb{R}, X) \). Moreover, the same argument works for separable complex Hilbert spaces.

THEOREM 2.5. \( MP(\mathbb{R}, X) \) is dense in \( C(\mathbb{R}, X) \).

PROOF. Case (i): \( X = \mathbb{C} \). It suffices to show that the annihilator of \( MP(\mathbb{R}) \) is \( \{0\} \). Let \( \mu \in M_c(\mathbb{R}) \) be such that \( \mu(MP(\mathbb{R})) = \{0\} \). In particular \( \mu(e^{i\lambda}) = \hat{\mu}(\lambda) = 0 \) for every \( \lambda \in \mathbb{C} \). Hence \( \mu = 0 \).

Case (ii): Let \( X \) be finite dimensional, \( X = \mathbb{C}^n \). Consider \( C(\mathbb{R}) \times C(\mathbb{R}) \times \cdots \times C(\mathbb{R}) \). This is a finite product of locally convex spaces. Hence it is a locally convex space in the product topology. It is easy to see that \( C(\mathbb{R}, X) \) is isomorphic to \( C(\mathbb{R}) \times \cdots \times C(\mathbb{R}) \) as locally convex spaces. Also \( MP(\mathbb{R}) \times MP(\mathbb{R}) \times \cdots \times MP(\mathbb{R}) \subseteq MP(\mathbb{R}, X) \) and \( MP(\mathbb{R}) \) is dense in \( C(\mathbb{R}, X) \). Thus it follows that \( MP(\mathbb{R}, X) \) is dense in \( C(\mathbb{R}, X) \).
Case (iii): $X$ is not finite dimensional. Consider the set $\text{Exp}(\mathbb{R}, X) = \{e^{i\lambda}x : \lambda \in \mathbb{C}, x \in X\}$. We show that the linear span of $\text{Exp}(\mathbb{R}, X)$ is contained in $MP(\mathbb{R}, X)$ and it is dense in $C(\mathbb{R}, X)$. Let $f(t) = e^{i\alpha t}x_1, g(t) = e^{i\beta t}x_2 \in \text{Exp}(\mathbb{R}, X)$ and $\alpha, \beta \in \mathbb{C}$. Choose $0 \neq x' \in X'$ such that $x'(x_1) = x'(x_2) = 0$ and $\mu_1, \mu_2 \in M(\mathbb{R})$ such that $e^{i\lambda t} \ast \mu_1 = 0 = e^{i\lambda t} \ast \mu_2$. Define $\mu(E) = (\mu_1 \ast \mu_2)(E)x'$, for every $E \in \mathcal{B}_\mathbb{R}$. Then $(\alpha f + \beta g) \ast \mu = 0$. To prove the denseness, let $\mu \in M(\mathbb{R}, X')$ be such that $\mu$ annihilates the linear span of $\text{Exp}(\mathbb{R}, X)$. Then $\hat{\mu}_\gamma(\lambda) = 0, \forall \lambda \in \mathbb{C}, \forall x \in X$. It follows that $\mu = 0$. This completes the proof.

We analyse next the problem of spectral analysis and spectral synthesis in $C(\mathbb{R}, X)$. Let $V$ be a closed translation invariant subspace of $C(\mathbb{R}, X)$. For $X = \mathbb{C}$, Schwartz [35] proved that $V$ contains exponential monomials and the linear span of exponential monomials in $V$ is dense in $V$. It is well known [17] that spectral synthesis fails for $\mathbb{R}^n, n > 1$. Further, it holds for certain locally compact abelian groups, namely for $\mathbb{Z}^n$ due to Lefranc [26] and discrete groups due to Gilbert [16, 15] and Elliott [12, 13]. However, nothing is known for vector valued functions. In this section, we extend Schwartz’s result for finite dimensional closed translation invariant subspace of $C(\mathbb{R}, X), X$ an arbitrary Banach space. For this we need the following lemmas.

**Lemma 2.6.** Let $v^1, v^2, \ldots, v^n \in X^n, v^i = (v^i_1, v^i_2, \ldots, v^i_n)$, be linearly independent. Then there exist $x^i_1, x^i_2, \ldots, x^i_n \in X'$ which satisfy

\[
x^i_1(v^i_1) + x^i_2(v^i_2) + \cdots + x^i_n(v^i_n) = 1,
\]

\[
x^i_1(v^i_1) + x^i_2(v^i_2) + \cdots + x^i_n(v^i_n) = 0,
\]

\[
\vdots
\]

\[
x^i_1(v^i_1) + x^i_2(v^i_2) + \cdots + x^i_n(v^i_n) = 0.
\]

**Proof.** Let $Y$ be the linear span of $\{v^2, v^3, \ldots, v^n\}$. Then $Y$ being a finite dimensional subspace of $X^n$ is closed. Since $v^1, v^2, \ldots, v^n$ are linearly independent, $v^1 \notin Y$. Thus by Hahn-Banach theorem, there exists $\Lambda \in (X^n)'$ such that $\Lambda(Y) = \{0\}$ and $\Lambda(v^1) = 1$. Clearly $\Lambda$ can be written as $\Lambda = (x^i_1, x^i_2, \ldots, x^i_n)$, where $x^i_j \in X'$ satisfy $\Lambda(x_1, x_2, \ldots, x_n) = x^i_1(x_1) + x^i_2(x_2) + \cdots + x^i_n(x_n)$. Therefore,

\[
x^i_1(v^i_1) + x^i_2(v^i_2) + \cdots + x^i_n(v^i_n) = \Lambda(v^i_1, 0, \ldots, 0) + \cdots + \Lambda(0, \ldots, 0, v^i_n)
\]

\[
= \Lambda((v^i_1, v^i_2, \ldots, v^i_n)) = 1.
\]

For every $i, 2 \leq i \leq n$,

\[
x^i_1(v^i_1) + x^i_2(v^i_2) + \cdots + x^i_n(v^i_n) = \Lambda(v^i_1, 0, \ldots, 0) + \cdots + \Lambda(0, \ldots, 0, v^i_n)
\]

\[
= \Lambda((v^i_1, v^i_2, \ldots, v^i_n)) = 0.
\]

This completes the proof of the lemma. 


For sets $A$ and $B$, let $\mathcal{F}(A, B)$ denote the set of all functions from $A$ to $B$. For a
set $E \subseteq V$, a vector space, let $LS(E)$ denote the linear span of $E$.

**Lemma 2.7.** Let $S$ be any set containing at-least $n$ points and $V$ be a vector
space over $\mathbb{C}$. Let $\{f_1, f_2, \ldots, f_n\} \subset \mathcal{F}(S, V)$. Then $\{f_1, f_2, \ldots, f_n\}$ is linearly independent in $\mathcal{F}(S, V)$ if and only if there exists $n$ distinct points $t_1, t_2, \ldots, t_n \in S$ such that $\{f_1, f_2, \ldots, f_n\}$ is linearly independent in $\mathcal{F}(\{t_1, t_2, \ldots, t_n\}, V)$.

**Proof.** We prove the straight implication by induction. Suppose that $\{f_1, f_2, \ldots, f_n\}$ is a linearly independent set in $\mathcal{F}(S, V)$. As $\{f_1\}$ is linearly independent, there exists $t_1 \in S$ such that $f_1(t_1) \neq 0$. Then $\{f_1\}$ is linearly independent on $\{t_1\}$. Thus the lemma is true when $n = 1$. If $f_1(t_1) = \alpha f_2(t_1)$, for some nonzero $\alpha \in \mathbb{C}$, choose $t_2 \in S$ such that $f_1(t_2) \neq \alpha f_2(t_2)$, which is possible, since $f_1, f_2, \ldots, f_n$ are linearly independent on $S$. Then it is easy to check that $\{f_1, f_2\}$ is linearly independent on $\{t_1, t_2\}$. If $f_1(t_1) \neq \alpha f_2(t_1)$ for any non zero scalar and $f_2(t_1) \neq 0$, then choose any $t_2 \neq t_1$. It is easy to see that $\{f_1, f_2\}$ is linearly independent on $\{t_1, t_2\}$. If $f_2(t_1) = 0$, then choose $t_2$ such that $f_2(t_2) \neq 0$. In this case also one can easily verify that $\{f_1, f_2\}$ is linearly independent on $\{t_1, t_2\}$. Assume that $\{f_1, f_2, \ldots, f_{n-1}\}$ is linearly independent on $\{t_1, t_2, \ldots, t_{n-1}\}$. If $\{f_1, f_2, \ldots, f_{n-1}, f_n\}$ is linearly independent on $\{t_1, t_2, \ldots, t_{n-1}\}$ then choose any $t_n$ which is different from $t_1, t_2, \ldots, t_{n-1}$. If $\{f_1, f_2, \ldots, f_{n-1}, f_n\}$ is linearly independent then there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that $\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_{n-1} f_{n-1} = f_n$ on $\{t_1, t_2, \ldots, t_{n-1}\}$. Since $\{f_1, f_2, \ldots, f_n\}$ is linearly independent on $S$, there exists $t_n \in S$ such that $\alpha_1 f_1(t_n) + \alpha_2 f_2(t_n) + \ldots + \alpha_{n-1} f_{n-1}(t_n) \neq f_n(t_n)$. It follows from this that $\{f_1, f_2, \ldots, f_n\}$ is linearly independent on $\{t_1, t_2, \ldots, t_n\}$. This proves the required claim.

The converse is trivial. \(\square\)

Using these lemmas we prove that every finite dimensional translation invariant subspace $V$ of $C(\mathbb{R}, X)$ includes an exponential and every element in $V$ is a finite sum of exponential monomials.

**Theorem 2.8.** Let $V$ be an $n$-dimensional translation invariant subspace of $C(\mathbb{R}, X)$. Then the following hold:

(i) There exist $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbb{C}$ and $m_1, m_2, \ldots, m_q \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_q = 0$, and $w_1, w_2, \ldots, w_q \in X$, not all zero, such that $e^{\lambda_i t} w_j \in V$, for $1 \leq j \leq q$.

(ii) There exist $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbb{C}$, $m_1, m_2, \ldots, m_q \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_q = 0$ and $x_1, x_2, \ldots, x_q \in X$ such that every $f \in V$ is of the form $f = \sum_{i=1}^{n} g_i x_i$, where each $g_i \in LS[1 e^{\lambda_i t}] : 0 \leq k \leq m_j - 1, 1 \leq j \leq q$.

(iii) There exist $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbb{C}$ and $m_1, m_2, \ldots, m_q \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_q = n$ such that every $f \in V$ is of the form $f = \sum_{j=1}^{q} \sum_{k=0}^{m_j-1} \alpha_{kj} t^k e^{\lambda_{kj} t} y_{kj}$, where $\alpha_{kj} \in \mathbb{C}$ and $y_{kj} \in X$ for $0 \leq k \leq m_j - 1, 1 \leq j \leq q$.
such that \( f \) is linearly independent on \( \{t_1, t_2, \ldots, t_n\} \). In view of (9), \( f_i(t_j) = A(s) f(t_j) \) for \( j = 1, 2, \ldots, n \). That is \( (f_i(s+t_j))_{i,j=1}^n = A(s)(f_i(t_j))_{i,j=1}^n \). Let \( v_i = (f_i(t_1), f_i(t_2), \ldots, f_i(t_n)) \). Then \( \{v_1, v_2, \ldots, v_n\} \) is a linearly independent subset of \( X^n \). By the Lemma 2.6 there exists \( x_{ij} \in X \) such that

\[
\sum_{k=1}^n (f_k(t_k), x_{ij}) = \delta_{ij}, \quad \text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}
\]

Thus we have

\[
(f_i(s+t_j))_{i,j=1}^n = A(s)(f_i(t_j))_{i,j=1}^n = A(s) x_{ij}. = (A(s) x_{ij}).
\]

The entries of the matrix obtained by multiplying the matrices on the left side of the above equation are continuous. This shows that \( s \mapsto A(s) \) is continuous from \( \mathbb{R} \) to \( BL(\mathbb{C}^n) \).

**Proof 2.** For every \( t \in \mathbb{R} \), define an operator \( T_t : V \rightarrow V \) by

\[
(T_t f)(s) := f(t+s), \quad \forall f \in V, \ s \in \mathbb{R}.
\]

Then \( T_t \in BL(V) \) and satisfies the following properties: For every \( s, t \in \mathbb{R} \)

i) \( T_s \circ T_t = T_{s+t} \);

ii) \( T_0 = I \);

iii) \( T_s \circ T_t = T_t \circ T_s \).

Let \( \{t_1, t_2, \ldots, t_n\} \) be as given by Lemma 2.7. Let \( \{K_n\}_{n \geq 1} \) be compact subsets of \( \mathbb{R} \) such that \( \bigcup_{n=1}^\infty K_n = \mathbb{R} \) with \( \{t_1, t_2, \ldots, t_n\} \subseteq K_1 \subseteq K_2 \subseteq \cdots \). To show the required claim we have to show that \( t \mapsto T_t \) is continuous in \( BL(V) \). We shall show first that \( t \mapsto T_t \) is continuous point-wise. Let \( s_n \rightarrow s \) as \( n \rightarrow \infty \). Now \( T_n(f) = f_s \) and \( T_t(f) = f_t \) for every \( f \in V \). Since \( f \) is uniformly continuous on compact sets, \( f_n \rightarrow f \) in \( C(\mathbb{R}, X) \). Therefore \( T_n \rightarrow T_s \) point-wise. To show that \( T_n \rightarrow T_s \) in \( BL(V) \), it is sufficient to show that for every \( m, \|T_n - T_s\|_{K_n} \rightarrow 0 \) as \( n \rightarrow \infty \), where
$\|T_{s_n} - T_s\|_{K_n} = \sup_{\|f\|_{K_n} \leq 1} \|T_{s_n}(f) - T_s(f)\|_{K_n}$. Let $\epsilon > 0$. Since $(f_1, f_2, \ldots, f_n)$ is a basis of $V$, for every $f$ in $V$, there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that $f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$. Also since $(f_1, f_2, \ldots, f_n)$ is linearly independent on $K_n$, $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ : $\|\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n\|_{K_n} \leq 1$ is bounded in $\mathbb{C}^n$, that is, there exists $M > 0$ such that $\|\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n\|_{K_n} \leq 1$ implies that $\|\alpha_1, \alpha_2, \ldots, \alpha_n\| \leq M$. Since $(f_1, f_2, \ldots, f_n)$ is equicontinuous, there exists a $\delta > 0$, with $\delta < 1$, such that whenever $t_1, t_2 \in s + K_m + [0, 1]$ with $|t_1 - t_2| < \delta$, $\|f_j(t_1) - f_j(t_2)\| < \epsilon/M$, for every $j = 1, 2, \ldots, n$. Choose $N \in \mathbb{N}$ such that $|s_n - s| < \delta$, whenever $n \geq N$. Then for every $f \in V$ with $\|f\|_{K_n} \leq 1$, for every $t \in K_m$, and $n \geq N$, we have

$$\|f_{s_n}(t) - f_s(t)\| = \|f(s_n + t) - f(s + t)\|$$

$$= \|\alpha_1 f_1 + \cdots + \alpha_n f_n)(s_n + t) - (\alpha_1 f_1 + \cdots + \alpha_n f_n)(s + t)\|$$

$$\leq |\alpha_1| \|f_1(s_n + t) - f_1(s + t)\| + \cdots + |\alpha_n| \|f_n(s_n + t) - f_n(s + t)\|$$

$$\leq \epsilon.$$

Thus $\|T_{s_n} - T_s\|_{K_n} \rightarrow 0$ as $n \rightarrow \infty$ for every $m$ and hence $T_{s_n} \rightarrow T_s$ in $BL(V)$ as $n \rightarrow \infty$. This completes the second proof of the claim.

Thus $A(s)$ satisfies the following properties:

(i) $s \mapsto A(s)$ is continuous.
(ii) $A(0) = I$.
(iii) $A(s + t) = A(s)A(t) = A(t)A(s)$.

Therefore, $s \mapsto A(s)$ is differentiable (refer [18]) and

$$A(s) = e^{A(0)s}.$$  

By virtue of equations (10) and (11),

$$f' = A'(0)f.$$  

This equation can be solved ([21]) and the solution is given by

$$f(t) = e^{A'(0)[x_1, x_2, \ldots, x_n]}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbb{C}$ be the eigen values of $A'(0)$ with multiplicities $m_1, m_2, \ldots, m_q$, respectively. Let the Jordan canonical form of $A'(0)$ be given by

$$BA'(0)B^{-1} = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_q \end{bmatrix}.$$
where $J_1, \ldots, J_q$ are the Jordan blocks of $A'(0)$, $B$ is an invertible matrix. This gives

$$e^{tA(0)} = B^{-1} \begin{bmatrix} B_1 & \cdots & \cdots & B_p \end{bmatrix} B,$$

where each $B_k$ is an $m_k \times m_k$ matrix given by

$$B_k = \begin{bmatrix} e^{i\lambda_1 t} & t e^{i\lambda_1 t} & \cdots & e^{i\lambda_1 t}(m_k - 1)! \\ 0 & e^{i\lambda_2 t} & \cdots & e^{i\lambda_2 t}(m_k - 2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\lambda_p t} \end{bmatrix}.$$ 

Thus $f(t) = C[x_1, x_2, \ldots, x_n]^T$, where $C = (c_{ij})$ and each $c_{ij} \in LS\{t^k e^{i\lambda_j t} : 0 \leq k \leq m_j - 1, 1 \leq j \leq q\}$, that is, for every $i$, $f_i(t) = \sum_{j=1}^{q} g_{ij}(t)x_j$, where $g_{ij} \in LS\{t^k e^{i\lambda_j t} : 0 \leq k \leq m_j - 1, 1 \leq j \leq q\}$. Hence every element $h$ of $V$ is of the form $h(t) = \sum_{j=1}^{n} g_{j}(t)x_j$, where each $g_{j} \in LS\{t^k e^{i\lambda_j t} : 0 \leq k \leq m_j - 1, 1 \leq j \leq q\}$. This proves (ii).

(iii) By the discussion above, each $f_i$ can be expressed as follows:

$$f_i = \sum_{j=1}^{q} \sum_{k=0}^{m_j - 1} t^k e^{i\lambda_j t} \beta_{kj}^i x_k^i.$$

Every $h \in V$ is of the form

$$h = \sum_{i=1}^{n} \alpha_i f_i = \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=0}^{m_j - 1} t^k e^{i\lambda_j t} \alpha_i \beta_{kj}^i x_k^i = \sum_{j=1}^{q} \sum_{k=0}^{m_j - 1} t^k e^{i\lambda_j t} \sum_{i=1}^{n} \alpha_i \beta_{kj}^i x_k^i = \sum_{j=1}^{q} \sum_{k=0}^{m_j - 1} t^k e^{i\lambda_j t} y_{kj}.$$

This proves (iii). For (i), $f_i = \sum_{j=1}^{q} \sum_{k=0}^{m_j - 1} t^k e^{i\lambda_j t} y_{kj}$. For every $j$ choose largest $k$ such that $y_{kj} \neq 0$, let it be $k_j$. We will show that $e^{i\lambda_j t}y_{kj}^j \in V$. To prove this, let $\mu \in M_c(\mathbb{R}, X)$ be such that $\mu(V) = \{0\}$. Then $f \star \mu = 0$ for every $f \in V$, since $V$ is translation invariant. Hence $f_i \star \mu = 0$, for every $i$. As $f_i \star \mu$ is a finite sum of complex valued exponential monomials and $\hat{\mu} y_{kj}^j(\lambda_j)$ is the coefficient of $e^{i\lambda_j t}$, $\hat{\mu} y_{kj}^j(\lambda_j) = 0$. This implies that $e^{i\lambda_j t}y_{kj}^j \in V$.

Corollary 2.9. Let $f \in C(\mathbb{R}, X)$. Then $\tau(f)$ is finite dimensional if and only if $f$ is a finite linear combination of exponential monomials in $C(\mathbb{R}, X)$. 

□
**Proof.** Suppose that \( \tau(f) \) is finite dimensional. Then it follows from the above theorem that \( f \) is a finite linear combination of exponential monomials. Conversely, suppose \( f \) is a finite linear combination of exponential monomials. Let \( f = \sum_{j=1}^{q} \sum_{k=0}^{m_j-1} \alpha_{jk} t^{k} e^{i\lambda_j} x_{jk} \). Then \( \tau(f) \subseteq L^{1} (\mathbb{R} ; \mathbb{C}) \). Therefore \( \tau(f) \) is finite dimensional.

**Remark.** (i) Some authors (see [14, 25]) define exponential polynomials to be functions of the form \( \sum_{j=1}^{n} f_j \), where \( f_j \) are exponential polynomials defined as in Definition 1.3. With this definition, our result states that every finite dimensional translation invariant subspace \( V \) of \( C(\mathbb{R}, X) \) is generated by exponential polynomials in \( V \).

(ii) Anselone and Korevaar [1] have proved that when \( X = \mathbb{C} \), \( V \subseteq C(\mathbb{R}) \) is finite dimensional if and only if \( V \) is the solution space of a homogeneous constant coefficient ordinary differential equation. This result is not true for arbitrary \( X \) which can be seen by the following examples.

**Example 3.** Let \( X \) be a separable infinite dimensional complex Hilbert space. Let \( \{e_{n}\} \) be a complete orthonormal basis. Consider the homogeneous ordinary differential equation with constant coefficient.

\[
a_{0} f + a_{1} f' + \cdots + a_{n} f^{(n)} = 0.
\]

Let \( \lambda_{1}, \lambda_{2}, \ldots, \lambda_{q} \) with multiplicities \( m_{1}, m_{2}, \ldots, m_{q} \) be the roots of the characteristic polynomial \( p(t) \). Then for every \( n \in \mathbb{N}, 0 \leq k \leq m_{j}, 1 \leq j \leq q \), \( t^{k} e^{i\lambda_{j}} e_{n} \) is a solution of the differential equation (13). Thus the solution space is not finite dimensional.

**Example 4.** Let \( X \) be a complex Banach space. Fix \( A \in \text{BL}(X) \). Consider the following differential equation \( du/dt = Au \). Then the solution space \( \{ u \in C(\mathbb{R}, X) : du/dt = Au \} = \{ e^{\lambda t} x : x \in X \} \) is a closed translation invariant subspace of \( C(\mathbb{R}, X) \). Further, it is finite dimensional if and only if \( X \) is finite dimensional.

Let \( \mu \in M_{\lambda}(\mathbb{R}, \mathbb{C}^{*}) \). In the case when \( X = \mathbb{C} \) it is known [35] that for a given \( \mu \) the linear span of exponential monomial solutions of the convolution equation \( f \ast \mu = 0 \) is dense in the space of all solutions. We extend this for \( X = \mathbb{C}^{*} \) as follows:

**Theorem 2.10.** Let \( f = (f_1, f_2, \ldots, f_n) \in C(\mathbb{R}, \mathbb{C}^{*}) \) satisfies the following:

(i) \( f_j \) is mean-periodic, for every \( 1 \leq j \leq n \);

(ii) \( \sigma(f_j) \cap \sigma(f_k) = \emptyset \) for \( j \neq k \).

Then \( \tau(f) \) contains exponential monomials and the linear span of exponential monomials in \( \tau(f) \) is dense in \( \tau(f) \).
PROOF. Clearly $\tau(f) \subseteq \tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)$. We show that these two sets are equal. Let $g \in \tau(f_1) \cap \tau(f_2)$. Then $\tau(g) \subseteq \tau(f_1) \cap \tau(f_2)$ and hence by Schwartz’s theorem, $\tau(f_1) \cap \tau(f_2) = \{0\}$. Thus $\tau(f_1) \cap \tau(f_j) = \{0\}$ for $i \neq j$. Let $\mu \in M_c(\mathbb{R}, \mathbb{X})$ be such that $\mu(\tau(f)) = \{0\}$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. Since $\tau(f)$ is translation invariant, $f \ast \mu = \sum_{j=1}^n f_j \ast \mu_j = 0$. Let $e^{i\lambda t}, te^{i\lambda t}, \ldots, t^{m-1}e^{i\lambda t} \in \tau(f)$ and $t^m e^{i\lambda t} \notin \tau(f_1)$. By Hahn-Banach theorem there exists a measure $\nu_1 \in M_c(\mathbb{R})$ such that $\nu_1(\tau(f_1)) = \{0\}$ for every $l \neq 1$ and $\nu_1(e^{i\lambda t}) \neq 0$. Therefore $f_l \ast \nu_1 = 0$, for $l \neq 1$. Now $f_1 \ast \mu_1 \ast \nu_1 = (f \ast \mu) \ast \mu_1 = 0$. Therefore $(\tilde{\mu}_1 \tilde{\nu}_1)(\lambda) = (\tilde{\mu}_1 \tilde{\nu}_1)'(\lambda) = \cdots = (\tilde{\mu}_1 \tilde{\nu}_1)^{(m_1-1)}(\lambda) = 0$. As $\tilde{\mu}_1(\lambda) = 0$ and $\tilde{\nu}_1(\lambda) = 0$. Also $(\tilde{\mu}_1 \tilde{\nu}_1)'(\lambda) = 0$ implies $\tilde{\mu}_1(\lambda)\tilde{\nu}_1(\lambda) + \tilde{\mu}_1(\lambda)\tilde{\nu}_1'(\lambda) = 0$. This implies $\tilde{\mu}_1(\lambda) = 0$. Similarly we can show that $\tilde{\mu}_1(\lambda) = \cdots = \tilde{\mu}_1^{(m_1-1)}(\lambda) = 0$. Thus $\lambda$ is a zero of $\tilde{\mu}_1$ with multiplicity at least $m_1$. This shows that $f_1 \ast \mu_1 = 0$. Similarly, $f_j \ast \mu_j = 0$ for every $j$. Thus $\mu(\tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)) = 0$. It follows that $\tau(f) = \tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)$. This completes the proof. \hfill \Box

**Corollary 2.11.** Let $X = \mathbb{C}^n$. Let $f = (f_1, f_2, \ldots, f_n) \in C(\mathbb{R}, X)$ and $\mu \in M_c(\mathbb{R}, X)$. Suppose that each $f_j$ is mean-periodic and $\sigma(f_j) \cap \sigma(f_k) = \emptyset$ for $j \neq k$. If $f \ast \mu = 0$, then $f$ is a finite linear combination of exponential monomials solutions.

**Proof.** Since spectral synthesis holds for $\mathbb{R}$, $LS(\sigma(f_j))$ is dense in $\tau(f_j)$, for every $j$. It is easy to see that $\sigma(f_1) \times \sigma(f_2) \times \cdots \times \sigma(f_n) \subseteq LS(E)$, where $E = \{t^j e^{i\lambda t} x : x \neq 0, t^j e^{i\lambda t} x \ast \mu = 0\}$. Thus $LS(E) = \tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)$. The required result follows from the Theorem 2.10. \hfill \Box

**Example 5.** (1) When $G = \mathbb{R}$ and $X = \mathbb{C}$, the notion of mean-periodic functions was introduced by Delastra in 1935 [5]. In [35] Schwartz gave an intrinsic characterization of mean-periodic functions: $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic if and only if $\tau(f)$, the closed translation invariant subspace of $C(\mathbb{R}, \mathbb{C})$ is proper. Clearly, for every $\lambda \in \mathbb{C}$, $f_\lambda(t) = e^{i\lambda t}, t \in \mathbb{R}$, is mean-periodic, $f \ast \mu = 0$ for $\mu = \delta_0 - e^{i\lambda} \delta_1$, where $\delta_0$ denote the Dirac measure on $\mathbb{R}$ at $x \in \mathbb{R}$. Schwartz [35] showed that if $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic with $f \ast \mu = 0$, then $f$ is a limit of finite linear combinations of functions of the type $f_\lambda(t) = t^k e^{i\lambda t}$, such that $f_\lambda \ast \mu = 0$. In Laird [22] it is shown that if $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic and $g$ is an exponential polynomial, that is, $g(t) = p(t)e^{i\lambda t}$, where $p(t)$ is a polynomial, then $fg$ is mean-periodic.

(2) Let $G$ be a compact abelian group. Then every character of $G$ is mean-periodic, as observed in Rana [33].

(3) For $X = \mathbb{C}$, mean-periodic functions on various locally compact groups have been analysed by various authors (see [2, 3, 5, 7, 10, 11, 17, 19, 20, 23, 24, 22, 29, 30, 36, 38, 37, 39]).
In general setting, even when $G = \mathbb{R}$ and $X$ is an arbitrary Banach space, nothing seem to be known.

Note. The following questions still remain unanswered:

1. Let $V$ be a closed translation invariant subspace of $C(\mathbb{R}, X)$. Does $V$ always include a monomial exponential? Is $V$ the closed linear span of the monomial exponentials in it?

2. The problem of finding solutions for $f \ast \mu = g$, for a given $\mu$ and $g$, seems to be much more difficult even for the case $G = \mathbb{R}$ and $X = \mathbb{C}$. Some particular situations are analysed in [31] and [32]. Another particular case is given in the next theorem.

Theorem 2.12. For a given $\mu \in M_c(\mathbb{R})$ and $g$ a finite sum of exponential polynomials in $C(\mathbb{R})$, there exists $f \in C(\mathbb{R})$ such that $f \ast \mu = g$.

Proof. First suppose that $g$ is an exponential polynomial. Let $g(t) = e^{i\lambda t} \sum_{k=0}^{n} a_k t^k$. Let $Z(\hat{\mu}) = \{ \lambda \in \mathbb{C} : \hat{\mu}(\lambda) = 0 \}$. We say

(i) $\lambda \in Z(\hat{\mu})$ is of multiplicity 0 if $\hat{\mu}(\lambda) \neq 0$.

(ii) $\lambda \in Z(\hat{\mu})$ of multiplicity $m \in \mathbb{N}$, if $\hat{\mu}(\lambda) = 0, \hat{\mu}'(\lambda) = 0, \ldots, \hat{\mu}^{(m-1)}(\lambda) = 0$ and $\hat{\mu}^{(m)}(\lambda) \neq 0$.

Let $m$ be the multiplicity of $\lambda \in Z(\hat{\mu})$. Define $f(t) = \sum_{k=0}^{n} b_k t^{m+k} e^{i\lambda t}$, where

$$b_n = \binom{(m+n)}{m} \mu^{(m)}(\lambda) a_n, \quad b_{n-1} = \left[ a_{n-1} - b_n \binom{(m+n+1)}{m+1} \mu^{(m+1)}(\lambda) \right] \binom{(m+n)}{m} \mu^{(m)}(\lambda) + \cdots,$$

$$b_0 = \left[ a_0 - b_1 \binom{(m+1)}{m} \mu^{(m+1)}(\lambda) \right] \binom{(m+n)}{m} \mu^{(m)}(\lambda) - \cdots$$

A simple computation of $f \ast \mu$ gives $f \ast \mu = g$. In the general case, suppose that $g = \sum_{j=1}^{p} g_j$, where $g_j(t) = p_j(t) e^{i\lambda_j t}$, for every $j$ and $\lambda_k \neq \lambda_j$ for $k \neq j$. Let $f_j$ be the exponential polynomial function corresponding to $g_j$ obtained as in the first case, that is, $f_j \ast \mu = g_j$. Then $f = \sum_{j=1}^{p} f_j$ is a solution of the given convolution equation.

3. Mean-periodic functions on $G = \mathbb{T}$

We shall consider integrals of $X$-valued functions with respect to scalar measures in the sense of Bochner integral, and the integrals of scalar valued continuous functions with respect to $X'$-valued measures in the sense similar to that of Bochner discussed in the last section.
**DEFINITION 3.1.** Let $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$. For every $n \in \mathbb{Z}$,

$$
\hat{f}(n) := \int_{\mathbb{T}} z^{-n} f(z) \, dz \quad \text{and} \quad \hat{\mu}(n) := \int_{\mathbb{T}} z^{-n} d\mu(z)
$$

are called the *$n$th-Fourier coefficient* of $f$ and $\mu$, respectively.

For $f \in C(\mathbb{T}, X)$, let $\tau(f)$ denote the closed translation invariant subspace generated by $f$.

**PROPOSITION 3.2.** $f \in C(\mathbb{T}, X)$ is mean-periodic if and only if $\tau(f) \neq C(\mathbb{T}, X)$.

**PROOF.** Follows from the fact that the dual of $C(\mathbb{T}, X)$ is $M(\mathbb{T}, X')$. \qed

**LEMMA 3.3.** For $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$, the following hold:

(i) $f \ast \mu$ is a uniformly continuous function on $\mathbb{T}$;

(ii) $(f \ast \mu) = (\hat{f}(n), \hat{\mu}(n))$.

**PROOF.** (i) Follows from the facts that $f$ is uniformly continuous, $\mu$ has finite variation and that $|(f \ast \mu)(z) - (f \ast \mu)(w)| \leq \int_{\mathbb{T}} \|f(z\xi) - f(w\xi)\| dV_{\mu}(\xi)$.

(ii) Since $\mathbb{T}$ is compact, $f$ is uniformly continuous on $\mathbb{T}$. Let $\epsilon_k > 0$ be such that $\epsilon_k \to 0$ as $k \to \infty$. Since the metric on $\mathbb{T}$ is invariant under rotation, there exist finite Borel partitions $P_k$ of $\mathbb{T} = \cup B_{k_j}$ such that if $z_{k_j}, w_{k_j} \in B_{k_j}$, then $\|f(z_{k_j}w) - f(w_{k_j}w)\| < \epsilon_k$ whenever $|w| = 1$. Now

$$
(f \ast \mu)(n) = \int_{\mathbb{T}} (f \ast \mu)(z) z^{-n} \, dz = \int_{\mathbb{T}} \int_{\mathbb{T}} f(zw) d\mu(w) z^{-n} \, dz
$$

$$
\quad \quad \quad \quad \quad \quad = \int_{\mathbb{T}} \lim_{k \to \infty} \left( \sum_{j} \langle f(z_{k_j}w_{k_j}), \mu(B_{k_j}) \rangle \right) z^{-n} \, dz.
$$

Since $f$ is continuous on $\mathbb{T}$, $f(\mathbb{T}) \subset B(0, r) = rB(0, 1)$ for some $r > 0$. We have

$$
\left| \sum_{j} \langle f(z_{k_j}w_{k_j}), \mu(B_{k_j}) \rangle \right| \leq \sum_{j} |\langle f(z_{k_j}w_{k_j}), \mu(B_{k_j}) \rangle| \leq \sum_{j} rV_{\mu}(B_{k_j}) \leq rV_{\mu}(\mathbb{T}) \leq rC.
$$

Applying dominated convergence theorem in (14) for the functions

$$
z \mapsto \sum_{j} \langle f(z_{k_j}w_{k_j}), \mu(B_{k_j}) \rangle z^{-n}$$
we obtain

\[(f \ast \mu)(n) = \lim_{k \to \infty} \int_{\Gamma} \sum_{j} \left\{ f(z u_{kj}), \mu(B_{kj}) \right\} z^{-n} dz\]

\[= \lim_{k \to \infty} \sum_{j} \int_{\Gamma} \left\{ f(z u_{kj}), \mu(B_{kj}) \right\} z^{-n} dz\]

\[= \lim_{k \to \infty} \sum_{j} \left( \int_{\Gamma} \left\langle \frac{z}{u_{kj}}, \mu(B_{kj}) \right\rangle d\mu(z) \right)\]

Now apply change of variable formula for the function \(z \mapsto \langle z^{-n} f(z u_{kj}), \mu(B_{kj}) \rangle\), to get

\[(f \ast \mu)(n) = \lim_{k \to \infty} \sum_{j} \left( \int_{\Gamma} \langle \frac{z}{u_{kj}}, \mu(B_{kj}) \rangle \hat{f}(n) \mid \mu(B_{kj}) \rangle \right)\]

\[= \left( \hat{f}(n), \lim_{k \to \infty} \sum_{j} \langle \frac{z}{u_{kj}}, \mu(B_{kj}) \rangle \mu(B_{kj}) \right) = (\hat{f}(n), \hat{\mu}(n)). \]

**COROLLARY 3.4.** For \(f \in C(\Gamma, X)\) and \(\mu \in M(\Gamma, X')\), \(f \ast \mu = 0\) if and only if \(\langle \hat{f}(n), \hat{\mu}(n) \rangle = 0\) for all \(n \in \mathbb{Z}\).

**PROOF.** Follows from Lemma 3.3 and the uniqueness of Fourier-Stieltjes coefficients of scalar valued functions on \(\Gamma\).

**PROPOSITION 3.5.** Let \(f \in C(\Gamma, X)\). Then \(\sigma(f) = \{ \alpha z^n \hat{f}(n) : \hat{f}(n) \neq 0 \text{ and } 0 \neq \alpha \in \mathbb{C} \} \).

**PROOF.** First we show that \(\{ \alpha z^n \hat{f}(n) : \hat{f}(n) \neq 0 \} \subseteq \sigma(f)\). Let \(\mu \in M(\Gamma, X')\) be such that \(\mu(\tau(f)) = 0\). Then \(f \ast \mu = 0\), since \(\tau(f)\) is translation invariant. Hence by Corollary 3.4, \(\langle \hat{f}(n), \hat{\mu}(n) \rangle = 0\) for every \(n\). Thus \(\mu(\alpha z^n \hat{f}(n)) = \alpha \langle \hat{f}(n), \hat{\mu}(n) \rangle = 0\), and by Corollary 3.4, \(\alpha z^n \hat{f}(n) \in \tau(f)\). Hence \(\alpha z^n \hat{f}(n) \in \sigma(f)\).

On the other hand, let \(z^n x \in \sigma(f)\). To show that \(x = \alpha \hat{f}(m)\) for some scalar \(\alpha\). Let \(x' \in X'\) be such that \(x'(\hat{f}(m)) = 0\). Let \(d\nu(z) = z^n x'dz\). Then

\[\hat{\mu}(n) = \begin{cases} x' & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}\]
Thus by Corollary 3.4, \( f \ast v = 0 \). Therefore \( \zeta_m x \ast v = 0 \) and hence \( \langle x, \hat{v}(m) \rangle = 0 \), that is, \( \langle x, x' \rangle = 0 \). Thus for \( x' \in X' \), \( \langle \hat{f}(m), x' \rangle = 0 \) implies \( \langle x, x' \rangle = 0 \). Therefore \( x = \alpha \hat{f}(m) \) for some \( \alpha \in \mathbb{C} \). This completes the proof.

**Proposition 3.6.** Let \( f \in C(\mathbb{T}, X) \). Then \( \sigma(f) = \emptyset \) if and only if \( f = 0 \).

**Proof.** By Proposition 3.5, it suffices to show that \( \hat{f}(n) = 0 \), for every \( n \in \mathbb{Z} \) if and only if \( f = 0 \). Using the uniqueness of Fourier coefficients for scalar valued functions we obtain, for every \( n \in \mathbb{Z} \) and \( x' \in X' \),

\[
\hat{f}(n) = 0 \Leftrightarrow \langle x', \hat{f}(n) \rangle = 0 \Leftrightarrow \left\langle x', \int_{\mathbb{T}} f(z)z^{-n} \, dz \right\rangle = 0 \Leftrightarrow \int_{\mathbb{T}} \langle x', f(z) \rangle z^{-n} \, dz = 0
\]

\[
\Leftrightarrow \langle x' \circ f \rangle(n) = 0 \Leftrightarrow x' \circ f = 0 \Leftrightarrow f = 0.
\]

**Theorem 3.7.** For a complex Banach space \( X \neq \mathbb{C} \) the following hold:

i) \( MP(\mathbb{T}, X) = C(\mathbb{T}, X) \).

ii) For every \( 0 \neq \mu \in M(\mathbb{T}, X) \), \( \{0\} \neq MP(\mu) \neq C(\mathbb{T}, X) \).

**Proof.** (i) Let \( f : \mathbb{T} \to X \) be a non zero continuous function. Then \( \hat{f}(n_0) \neq 0 \) for some \( n_0 \). Chose \( x' \in X' \) such that \( x' \neq 0 \) and \( \langle x', \hat{f}(n_0) \rangle = 0 \). Define \( \mu(E) := \left( \int_{E} \zeta^n \, dz \right) x' \), for every \( E \in \mathcal{B}_T \). Then \( \mu \in M(\mathbb{T}, X') \) and

\[
\hat{\mu}(n) = \begin{cases} 
  x' & \text{if } n = n_0; \\
  0 & \text{if } n \neq n_0.
\end{cases}
\]

Thus \( (f \ast \hat{\mu})(n) = \langle \hat{f}(n), \hat{\mu}(n) \rangle = 0 \), for every \( n \in \mathbb{Z} \). Hence it follows from Corollary 3.4, \( f \ast \mu = 0 \).

(ii) Let \( 0 \neq \mu \in M(\mathbb{T}, X) \). Then \( \hat{\mu}(n_0) \neq 0 \) for some \( n_0 \). Let \( 0 \neq x \in X \) be such that \( \langle \mu(n_0), x \rangle = 0 \), and \( y \in X \) be such that \( \langle \mu(n_0), y \rangle \neq 0 \). Define \( f, g : \mathbb{T} \to X \), by \( f(z) = \zeta^n x \) and \( g(z) = \zeta^n y \). Then

\[
\hat{f}(n) = \begin{cases} 
  x & \text{if } n = n_0; \\
  0 & \text{if } n \neq n_0
\end{cases}, \quad \text{and} \quad \hat{g}(n) = \begin{cases} 
  y & \text{if } n = n_0; \\
  0 & \text{if } n \neq n_0
\end{cases}
\]

Therefore, \( (\hat{f}(n), \hat{\mu}(n)) = 0 \) for all \( n \in \mathbb{Z} \) and \( (g \ast \mu)(n_0) = \langle \hat{g}(n_0), \hat{\mu}(n_0) \rangle = 0 \). Thus \( f \) is mean-periodic with respect to \( \mu \) and \( g \) is not mean-periodic with respect to \( \mu \).

**Remark.** (1) Theorem 3.7 (i) is not true when \( X = \mathbb{C} \). For instance, the function \( f : \mathbb{T} \to \mathbb{C} \) defined by \( f(z) := \sum_{n=0}^{\infty} a_n z^n \), \( z \in \mathbb{T} \), where \( a_n \in \mathbb{C} \), \( a_n \neq 0 \) for every \( n \) and \( \sum_{n=0}^{\infty} |a_n| < \infty \) is not mean-periodic.
(2) Let $G$ be a locally compact abelian group and $X$ a complex Banach space. A function $f \in C(G, X)$ is said to be \textit{almost periodic} if the set of all translates of $f$ is relatively compact in $C(G, X)$. Every $f \in C(\mathbb{T}, X)$ is almost periodic and if $X \neq \mathbb{C}$, then every $f \in C(\mathbb{T}, X)$ is mean-periodic. When $X = \mathbb{C}$, there are complex valued continuous functions on the circle group $\mathbb{T}$ which are not mean-periodic.

We have the following result for spectral analysis and spectral synthesis for $\mathbb{T}$.

**Theorem 3.8.** The following hold:

(i) Let $x \in X$, $x \neq 0$, and $n_0 \in \mathbb{Z}$. Then $\tau(z^{n_0}x)$, the closed translation invariant subspace generated by $z^{n_0}x$, does not contain any non-zero proper closed translation invariant subspace of $C(\mathbb{T}, X)$.

(ii) Every non-zero closed translation invariant subspace $V$ of $C(\mathbb{T}, X)$ contains an exponential, that is, spectral analysis holds in $C(\mathbb{T}, X)$.

(iii) The linear span of the exponentials in every closed translation invariant subspace $V$ of $C(\mathbb{T}, X)$ is dense in $V$, that is, spectral synthesis holds in $C(\mathbb{T}, X)$.

**Proof.** (i) Let $V_1$ be a non-zero closed translation invariant subspace of $C(\mathbb{T}, X)$ such that $V_1 \subseteq \tau(z^{n_0}x)$. Then for $f \in \tau(z^{n_0}x)$, $\hat{f}(n_0) = cx$ for some $0 \neq c \in \mathbb{C}$ and $\hat{f}(n) = 0$ if $n \neq n_0$. To show $V_1 = \tau(z^{n_0}x)$, let $\mu \in M(\mathbb{T}, X')$ be such that $\mu(V_1) = \{0\}$. Then $\langle \hat{\mu}(n), x \rangle$ for every $n$. In particular $\langle \hat{\mu}(n_0), x \rangle$ and hence $\mu(V_1) = \{0\}$. Hence $V_1 = \tau(z^{n_0}x)$.

(ii) Choose $n_0 \in \mathbb{Z}$ and $f \in V$ such that $\hat{f}(n_0) \neq 0$. We will show that $z^{n_0} \hat{f}(n_0) \in V$. For, let $\mu \in M(\mathbb{T}, X')$ be such that $\mu(V) = \{0\}$. Since $V$ is translation invariant and $\mu(V) = \{0\}$, $f \star \mu = 0$. This implies $\langle \hat{f}(n_0), \hat{\mu}(n_0) \rangle = 0$. Thus $z^{n_0} \hat{f}(n_0) \star \mu = 0$. Hence $z^{n_0} \hat{f}(n_0) \in V$.

(iii) Let $V$ be closed translation invariant subspace of $C(\mathbb{T}, X)$. Let $V_0$ be the closed linear span of $z^n \hat{f}(n)$, $f \in V$. Then by (ii), $V_0 \subseteq V$. Let $f \in V$. Let $\mu \in M(\mathbb{T}, X')$ such that $\mu(V_0) = \{0\}$. Then $\langle f(n), \hat{\mu}(n) \rangle = 0$, for every $n \in \mathbb{Z}$. Thus $f \star \mu = 0$. Therefore, $\mu(f) = 0$. \hfill \Box

**Corollary 3.9.** For $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$, the following are equivalent:

(i) $f \star \mu = 0$.

(ii) $f$ is a limit of finite linear combinations of functions $z^n x$ which satisfy the equation $z^n x \star \mu = 0$.

**Proof.** First observe that for a given $\mu$, $MP(\mu) = \{f \in C(\mathbb{T}, X) : f \star \mu = 0\}$ is a closed translation invariant subspace of $C(\mathbb{T}, X)$. The result follows from Theorem 3.8 (iii). \hfill \Box
4. Some results for general groups

As mentioned earlier, problem of analysing mean-periodic functions, the problem of spectral analysis and spectral synthesis seems difficult to answer for general groups. However, it is not difficult to show that if $G$ is compact and $X = \mathbb{C}$ then every nontrivial closed translation invariant subspace $V$ of $C(K, \mathbb{C})$ includes exponentials and the linear span of exponentials in $V$ is dense in it. Hence every mean-periodic (scalar valued) function on a compact group is a limit of finite linear combination of exponentials.

For $G$ arbitrary locally compact abelian, and $X = \mathbb{C}$ we have the following: recall, $\Omega = \{ \omega : G \rightarrow \mathbb{C}^+ : \omega \in C(G) \text{ and } \omega(g_1 + g_2) = \omega(g_1)\omega(g_2) \}$.

**Theorem 4.1.**

(i) Every $\omega \in \Omega$ is mean-periodic.

(ii) Let $G$ be an infinite locally compact $T_1$ abelian group. Then every exponential polynomial on $G$ is mean-periodic.

(iii) Let $M_P(G)$ be the set of all mean-periodic functions on $G$. Then $M_P(G)$ is dense in $C(G)$ if and only if $G$ is non-trivial.

**Proof.**

(i) Clearly, every translate $O_x$ of $\omega$ is a constant multiple of $\omega$, and hence every finite linear combination of translates of $\omega$ is a constant multiple of $\omega$. Therefore the closed translation invariant subspace $\tau(\omega)$ is a one dimensional subspace of $C(G)$. Thus $\tau(\omega) \neq C(G)$ if $G$ is non-trivial.

(ii) Let $f$ be an exponential polynomial on $G$,

$$f(g) := \left( \sum_a c_a a_1(g)^{\alpha_1} a_2(g)^{\alpha_2} \cdots a_m(g)^{\alpha_m} \right) \omega(g),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbb{N}, c_a$ are complex constants and $a_1, \ldots, a_m$ are additive functions. Let $V = LS(a_1(g)^{\beta_1} a_2(g)^{\beta_2} \cdots a_m(g)^{\beta_m} \omega(g) : \beta_i \in \mathbb{Z}, \beta_i \leq \alpha_i$ for $1 \leq j \leq m \}$.

(iii) Suppose that $G$ is finite, $G = \{ g_1, g_2, \ldots, g_n \}$. Let $f \in C(G)$ and $\mu \in M_1(G)$.

Let $\mu(g_i) = c_i$. Then $f \ast \mu = 0$ for a non-trivial $\mu$ if and only if

$$\begin{vmatrix}
  f(g_1 - g_1) & f(g_1 - g_2) & \cdots & f(g_1 - g_n) \\
  f(g_2 - g_1) & f(g_2 - g_2) & \cdots & f(g_2 - g_n) \\
    \vdots & \vdots & \ddots & \vdots \\
  f(g_n - g_1) & f(g_n - g_2) & \cdots & f(g_n - g_n)
\end{vmatrix} = 0.$$

The columns of the above matrix are permutations of $[f(g_1), f(g_2), \ldots, f(g_n)]$. Thus $f$ is mean-periodic if and only if $(f(g_1), f(g_2), \ldots, f(g_n))$ is a root of some
fixed polynomial $P$ in the variables $z_1, z_2, \ldots, z_n$. The roots of this polynomial $P$ form a closed set $Z(P)$ in $\mathbb{C}^n$ of $2n$-dimensional Lebesgue measure zero. Therefore $Z(P)$ is not dense in $\mathbb{C}^n$. But $MP(G) = Z(P)$. Hence $MP(G)$ is not dense in $C(G)$.

Conversely, suppose that $G$ is not finite. Let $EP(G)$ be the set of all exponential polynomials in $C(G)$. By (ii), $EP(G) \subseteq MP(G)$, that is, $\Gamma \subseteq \Omega \subseteq EP(G) \subseteq MP(G)$. Moreover $\Omega$ separates points of $G$. Since the pointwise product of finite number of exponentials is again an exponential, it is easy to see that product of two exponential polynomials $f$ and $g$ is a finite sum of exponential polynomials and hence $\tau(fg)$ is finite dimensional. Therefore the algebra $A(EP(G))$, generated by $EP(G)$, is contained in $MP(G)$, that is, $A(EP(G)) \subseteq MP(G)$. Hence by Stone-Weierstrass theorem ([9]) $A(EP(G))$ is dense in $C(G)$. Since $A(EP(G)) \subseteq MP(G)$, $MP(G)$ is dense in $C(G)$.

**Corollary 4.2.** If $G$ is a finite $T_1$ topological abelian group, then $\{0\} \neq MP(G) \neq C(G)$.

**Lemma 4.3.** Let $G$ be a locally compact abelian group having no nontrivial compact subgroups. Let $\hat{G}$ be the dual group of $G$. Then for $\mu \in M_c(G)$, $\lambda_\mu(\{y \in G : \hat{\mu}(y) = 0\}) = 0$.

**Proof.** Refer [6].

**Theorem 4.4.** If $G$ does not have compact elements, then $\{0\} \neq MP(G) \neq C(G)$.

**Proof.** Let $f \in C(G)$ be compactly supported. By Lemma 4.3, $f$ is not mean-periodic. Thus $MP(G) \neq C(G)$.

As we have pointed earlier, the problem of spectral synthesis does not hold for every closed translation invariant subspace $V$ of $C(\mathbb{R}^2, \mathbb{C})$. However, with some conditions on $V$ this is true. First we prove the following lemma.

**Lemma 4.5.** The following hold:

(i) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct complex numbers and $m_1, m_2, \ldots, m_n \in \mathbb{N}$. Then the set $\{e^{i\lambda_j t}, t e^{i\lambda_j \beta_j} : 1 \leq j \leq n\} \subseteq C(\mathbb{R})$ is linearly independent over $\mathbb{C}$.

(ii) Let $\lambda_1, \lambda_2, \ldots, \lambda_n; \eta_1, \eta_2, \ldots, \eta_n$ be complex numbers and for $1 \leq j, k, l \leq n$, $\alpha_j, \beta_j$ be non-negative integers. Then $\{t^{\alpha_j} e^{i\lambda_j t} + b e^{i\eta_j t} : 1 \leq l, j \leq n\}$ is a linearly independent subset of $C(\mathbb{R}^2)$ over $\mathbb{C}$ if $(\lambda_j, \eta_j) \neq (\lambda_k, \eta_k)$ or $(\alpha_j, \beta_j) \neq (\alpha_k, \beta_k)$.

**Proof.** (i) Without loss of generality, we may assume that

$$\text{Im}(\lambda_n) = \max_{1 \leq j \leq n} \text{Im}(\lambda_j),$$
where Im denotes the imaginary part of a complex number. Then \( \text{Im}(\lambda_n) - \text{Im}(\lambda_i) > 0 \) for \( 1 \leq j \leq n - 1 \). Now for \( a_{ij} \in \mathbb{C} \),

\[
\sum_{j=1}^{n} \left( a_{0j} e^{i\beta_j t} + a_{1j} t e^{i\beta_j t} + \cdots + a_{nj} t^m e^{i\beta_j t} \right) = 0 \implies \\
\sum_{j=1}^{n-1} \left( a_{0j} e^{i(\beta_j - \lambda_n) t} + a_{1j} t e^{i(\beta_j - \lambda_n) t} + \cdots + a_{nj} t^m e^{i(\beta_j - \lambda_n) t} \right) + p_n(t) = 0,
\]

where \( p_n(t) = a_{0n} + a_n t + \cdots + a_{nm} t^m = a_{nm} (t - \beta_1)(t - \beta_2) \cdots (t - \beta_n) \), for some \( \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{C} \). Now as \( t \to -\infty \), \( t^k e^{i(\beta_j - \lambda_n)} \to 0 \) for every \( j \neq n \). This implies \( a_{nm} = 0 \), since as \( t \to -\infty \), \( p_n(t) \not\to 0 \) if \( a_{nm} \neq 0 \). Similarly by repeating the same argument one can easily show that \( a_{ij} = 0 \) for all \( i, j \).

(ii) Case (i): \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are distinct. For \( a_{ij} \in \mathbb{C} \),

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( a_{ij} t_1^{p_i} t_2^{p_j} e^{i(\omega_i t_1 + \omega_j t_2)} \right) = 0 \implies \sum_{i=1}^{n} \sum_{j=1}^{n} \left( (a_{ij} t_2^{p_j} e^{i(\omega_j t_1 + \omega_j t_2)} t_1^{a_i} e^{i\lambda_j t_1}) \right) = 0.
\]

By (i), this implies \( a_{ij} = 0 \) for all \( i, j \).

Case (ii): \( \lambda_i = \lambda_j \) for some \( i \) and \( j \). In this case rearrange the terms of the above expression by collecting the distinct exponential monomials in \( t_1 \). By the hypothesis, the coefficients of the exponential monomial in \( t_1 \) are finite linear combination of exponential monomials in \( t_2 \). By applying (i) twice, namely, first \( t_1 \) variable and then \( t_2 \) variable we get \( a_{ij} = 0 \) for all \( i, j \).

**Theorem 4.6.** Let \( V \) be a closed translation invariant subspace of \( C(\mathbb{R}^2) \) satisfying any one of the following conditions:

(i) \( V \) is finite dimensional.

(ii) \( V \) is rotation invariant.

(iii) \( V = \tau_\mu := \{ f \in C(\mathbb{R}^2) : f \ast \mu = 0 \} \) for some \( \mu \in M_1(\mathbb{R}^2) \).

Then \( V \) contains an exponential.

**Proof.** Case (i): \( V \) is finite dimensional. Let \( f \in V \) and \( f \neq 0 \). By Theorem 1.9, \( f \) is of the form \( f = \sum_{j=1}^{n} p_j (t_1, t_2) e^{i(\lambda_j t_1 + \eta_j t_2)} \), where \( p_j \) is a non-zero polynomial in \( t_1, t_2 \) and \( (\lambda_j, \eta_j) \neq (\lambda_k, \eta_k) \) for \( j \neq k \). Let \( \mu \in M_1(\mathbb{R}^2) \) be such that \( \mu(V) = \{0\} \).

We show that \( \mu(e^{i(\lambda_j t_1 + \eta_j t_2)}) = 0 \). Since \( V \) is translation invariant, \( f \ast \mu = 0 \). Write \( f \) as a linear combination of elements in \( \{ t_1^{\omega_1} t_2^{\omega_2} e^{i(\lambda_j t_1 + \eta_j t_2)} : 1 \leq 1, j \leq n \} \). Let \( c_{k_1} t_1^{\omega_1} t_2^{\omega_2} e^{i(\lambda_1 t_1 + \eta_1 t_2)}, c_{k_2} t_1^{\omega_3} t_2^{\omega_4} e^{i(\lambda_1 t_1 + \eta_1 t_2)}, \ldots, c_{k_n} t_1^{\omega_n} t_2^{\omega_n} e^{i(\lambda_1 t_1 + \eta_1 t_2)} \) be the terms containing \( e^{i(\lambda_1 t_1 + \eta_1 t_2)} \) and the largest degree term of \( t_1 \), namely \( t_1^{\omega_n} \), where \( c_{k_1}, c_{k_2}, \ldots, c_{k_n} \) are non-zero scalars. Also, \( f \ast \mu \) has the same representation and the terms containing \( t_1^{\omega_n} e^{i(\lambda_j t_1 + \eta_j t_2)} \) are \( c_{k_1} \hat{\mu}(\lambda_1, \eta_1) t_2^{\omega_n} t_1^{\omega_1} e^{i(\lambda_1 t_1 + \eta_1 t_2)}, c_{k_2} \hat{\mu}(\lambda_1, \eta_1) t_2^{\omega_n} t_1^{\omega_3} e^{i(\lambda_1 t_1 + \eta_1 t_2)}, \ldots, 

\]
Since $f \ast \mu = 0$ and $c_{k_j} \neq 0$, $\hat{\mu}(\lambda_j, \eta_j) = 0$, by Lemma 4.5. Therefore $\mu(e^{i(\lambda_j, \eta_j)z}) = 0$. Thus $e^{i(\lambda_j, \eta_j)z} \in V$.

Cases (ii) and (iii): $V$ is rotation invariant, or $V = \tau_\mu$. By Theorem 1.7 and Theorem 1.8, $V$ contains an exponential polynomial. It follows easily from the proof of (i) that $V$ contains an exponential.

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