Technical Papers

A Poincaré duality in K-theory

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Poincaré duality is a fundamental result that relates the homology and cohomology groups of manifolds. For M a compact orientable manifold of dimension n, it states that we have an isomorphism

$$H_k(M) \cong H^{n-k}(M)$$

for each $k \in \{0, \ldots, n\}$, where we may take, for example, coefficients in $\mathbb{Z}$. If we take instead coefficients in $\mathbb{R}$, Poincaré duality implies that, in order to determine the de Rham cohomology of an oriented compact manifold, it suffices to calculate only the first $\lceil \frac{n}{2} \rceil$ groups. The isomorphism can be realised via the cap product, a bilinear map

$$\cap\colon H^q(M) \times H_p(M) \to H_{p-q}(M), \qquad p \geq q,$$

in the following way. Orientability of a compact manifold means that there exists an element $[M] \in H_n(M)$, called the fundamental class, with the property that when the second argument above is restricted to $[M]$, $\cap$ gives an isomorphism of abelian groups. That is, we have

$$\cap\colon H^q(M) \times [M] \xrightarrow{\sim} H_{n-q}(M).$$

A simple example of a compact non-orientable manifold where Poincaré duality fails to hold (over $\mathbb{Z}$) is the Klein bottle K^2, whose homology groups are $H_0(K^2) = \mathbb{Z}$, $H_1(K^2) = \mathbb{Z} \oplus \mathbb{Z}_2$, and $H_2(K^2) = 0$.

In the setting of complex K-theory, an extraordinary cohomology theory, a version of Poincaré duality also holds for manifolds that satisfy an analogous notion of orientability. Here, the cohomology groups $H^k(M)$ are replaced by the complex topological K-theory groups $K^i(M)$, with $i \in \{0, 1\}$, while the homology groups $H_k(M)$ are replaced by the K-homology groups $K_i(M)$. The notion corresponding to that of an orientable manifold is that of a so-called Spin^c manifold.

The group $K^0(M)$ is constructed using complex vector bundles, subject to an equivalence relation defined by stable isomorphism. The group $K_0(M)$ is constructed from certain abstract elliptic operators acting on sections of complex vector bundles over M, subject to a certain equivalence relation [3]. Recall that ellipticity of an operator is defined by invertibility of its symbol; a familiar example of an elliptic operator is the Laplacian on $\mathbb{R}^n$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.

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There is also a K-theoretic analogue of the cap product,

$$\cap\colon K^q(M) \times K_p(M) \to K_{p-q}(M),$$

where the index $p - q$ is taken mod 2. The notion of Spin^c is strictly stronger than that of orientability, as we explain now. Note that one way to characterise orientability is to say that the restriction of the tangent bundle TM over any embedded loop $S^1 \hookrightarrow M$ is trivialisable. A manifold M of even dimension greater than or equal to 4 is said to be Spin^c if it is orientable and the restriction of TM over any embedded 3-sphere $S^3 \hookrightarrow M$ has a complex structure.

A Spin^c-structure on a manifold makes it possible to construct an operator that plays an analogous role to that of the fundamental class in homology, called the Spin^c-Dirac operator, D. Suppose, for simplicity, that M is an even-dimensional compact Spin^c-manifold. Then the elliptic operator D defines a class $[D] \in K_0(M)$. $[D]$ plays the role of $[M]$ in the sense that restriction on the second factor of $\cap$ produces the following isomorphism of abelian groups:

$$\cap\colon K^0(M) \times [D] \xrightarrow{\sim} K_0(M),$$
$$[E] \cap [D] \mapsto [D_E].$$

Here E denotes a complex vector bundle over M and D_E the Spin^c-Dirac operator on M twisted by E. This is the analogue of Poincaré duality in the setting of K-theory when the manifold M is Spin^c.

In joint work with my supervisors Professor Mathai Varghese and Dr Hang Wang [1], we establish a Poincaré duality for the equivariant version of K-theory — that is, where one takes into consideration the action of a Lie group G on a (possibly non-compact) manifold X and compatible actions of G on complex vector bundles over it. For a fixed compact Lie group G, the topological K-theory group is replaced by a group $K^0_G(X)$, defined using G-equivariant vector bundles; an account of this theory can be found in [6]. The equivariant theory of K-homology, denoted $K^G_0(X)$ (where G may be non-compact), can be found in [3]. Elements in this group are represented by abstract G-invariant elliptic operators on the manifold subject to a certain equivalence relation.

The setting of our result is as follows. Suppose G is a Lie group with finitely many connected components, acting properly on a smooth even-dimensional manifold X (not necessarily compact) with a G-equivariant Spin^c-structure, and that the orbit space is compact. The action is said to be proper if the inverse image of any compact set under the map

$$\mu\colon G \times X \to X \times X,$$
$$(g, x) \mapsto (x, g \cdot x)$$

is compact. For example, the action of a compact Lie group on any manifold is necessarily proper. On the other hand, the action of $\mathbb{Z}$ on S^1 by irrational rotations is not proper, although it is free.

Under these assumptions, we establish the following equivariant version of Poincaré duality:

$$K^0_G(X) \cong K^G_0(X),$$
$$[E] \mapsto [D] \cap [E] := [D_E],$$

where E is a G-equivariant complex vector bundle, D is the G-invariant Spin^c-Dirac operator on X, and D_E a twisted operator. This map is entirely similar to that in the non-equivariant version of Poincaré duality. Now when the Lie group G is non-compact, the elements of $K^0_G(X)$ cannot always be represented by finite-dimensional G-equivariant vector bundles [4]. However, it can be shown that when G has only finitely many connected components, finite-dimensional vector bundles are enough [5], so that the above isomorphism makes sense.

The Poincaré duality in [1] is the first such result in equivariant K-theory for non-compact groups acting on non-compact manifolds. It generalises a previous result of Kasparov [2], which covers the case of compact G and compact X, to a much larger class of Lie groups. The requirement that X be even-dimensional, which we invoked above in order to simplify notation, can be removed without much difficulty. In addition, in the same paper [1], we establish a more general Poincaré duality where the assumption that M be Spin^c is dropped.

References

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