The Izergin–Korepin model at roots of unity

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Among the models of interacting classical statistical mechanics the Yang–Baxter (YB) integrable systems play a special role [1]. The central model in the theory of YB integrable systems is the six vertex model. The model under our consideration is the Izergin–Korepin (IK) nineteen vertex model, which can be viewed as a generalization of the six vertex model [2]. The IK model is defined on a square lattice with states living on the edges of the lattice. There are three possible states at each edge. If the edge is horizontal the three states are: right-arrow, left-arrow and empty edge; if the edge is vertical the states are: up-arrow, down-arrow and empty edge. Additionally we require that at each vertex the number of entering arrows must be equal to the number of exiting arrows. With these conditions we obtain the nineteen possible vertex configurations Figure 1. The first six vertices on Figure 1 are the vertices of the six vertex model.

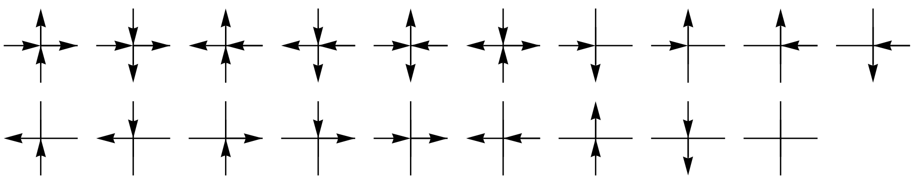

Figure 1. The nineteen vertices of the IK model

Placing the nineteen vertices on a square lattice such that the arrows represent a continuous flow gives a configuration of the IK model Figure 2. Each vertex enters a configuration with a specific Boltzmann weight which is determined by the so called Yang–Baxter equation. The weight of a configuration is then given by the product of all the weights of the individual vertices. Summing the weights of all configurations in a given domain with fixed boundary conditions gives the corresponding partition function.

This model is interesting from various perspectives. One of its main purposes is to provide a convenient framework for studying the spectrum of a one dimensional

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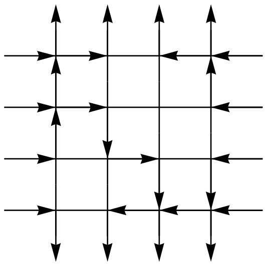

Figure 2. A typical configuration on a 4×4 domain

interacting quantum many body system defined by the Izergin–Korepin Hamiltonian H_{IK} [2]. The interaction is encoded by a parameter called q. The eigenstates of the Hamiltonian H_{IK} at certain interaction regimes are related to interesting statistical physics models. These are a model of percolation (when $q = e^{i\pi/3}$) and a model of polymer chains (when $q = e^{i\pi/4}$). Both models can be described using a gas of loops. In the case of percolation the loops separate the percolating and non-percolating regions. In the second case, the interaction is chosen such that only one loop is allowed. This loop corresponds to the polymer.

Let us briefly introduce the loop model and show its relation to the IK vertex model. The loop model is defined on a square lattice. Each face of the lattice contains a link that connects any two different edges of that face or two links that connect two edges in a nonintersecting manner. Edges which are linked are called occupied edges, otherwise they are called unoccupied. Since neighbouring faces share an edge, its occupation on the two faces must agree. In this way we obtain a collection of loops on a square lattice (first picture on Figure 3). To map this loop model to the vertex model we need to orient the loops by assigning an arrow to the loops. Each face of the oriented loop model can be turned into a vertex on the dual lattice. An example of this map is given on Figure 3.

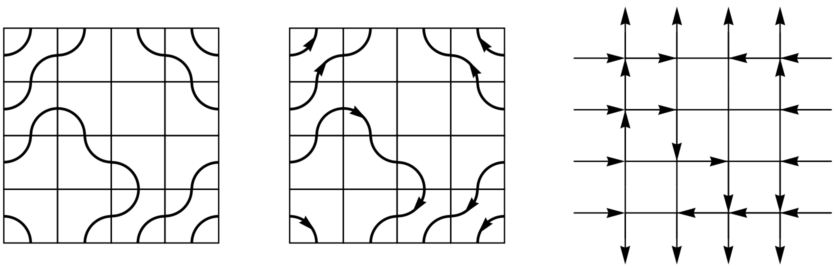

Figure 3. Mapping a configuration of the loop model to a configuration of the vertex model

Let us take as an application a model of the polymer chain on an infinite cylinder C with the circumference L. Consider a square lattice on C. The vertices of the lattice have coordinates (x, y), with $x = 0, 1, \ldots, L$ and $y = -\infty, \ldots, -1, 0, 1, \ldots, \infty$. Cutting the cylinder in half along the coordinate $y = 0$, we get two 'half-infinite' cylinders C^+ $(y \geq 0)$ and C^- $(y \leq 0)$. The ground state of the loop model Ψ is given as a linear combination of loop states on the half-infinite cylinder C^+. The half-infinite cylinder C^+ has a free boundary, the circle $(x, y = 0)$, on which loops can end and form a connectivity pattern $|\pi\rangle$. Hence we can group all configurations according to their loop connectivity $|\pi\rangle$ on the free boundary and write the ground state as a vector in the space of all connectivities

$$\Psi = \sum_{\pi} \psi_\pi |\pi\rangle.$$

The components ψ_π are given by the combined weight of all configurations which have the connectivity π on the free boundary. Equally, we can define the dual state $\bar{\Psi}$ which lives on the other half-infinite cylinder C^-; its components are denoted by $\bar{\psi}_\pi$ and the connectivity patterns label the basis vectors $\langle\pi|$. The scalar multiplication of the two states Ψ and $\bar{\Psi}$ can be considered as joining C^+ and C^- along the free boundary $(x, y = 0)$ and it represents a sum over states which live on the full cylinder C. Let us take the components ψ_π and $\bar{\psi}_\pi$ and form the following sum

$$Z_L = \sum_{\substack{\alpha,\beta: \\ \langle\alpha|\beta\rangle=\circ}} \psi_\alpha \bar{\psi}_\beta, \tag{1}$$

where the notation $\langle\alpha|\beta\rangle = \circ$ means that we sum over all α and β which correspond to the connectivities that form a single loop (see Figure 4). The restriction to the interaction regime $q = e^{i\pi/4}$ ensures that there are no other loops on the cylinder except from the one formed by the patterns α and β. Therefore, the quantity Z_L is the partition function of the polymer chain on a infinite cylinder with circumference L.

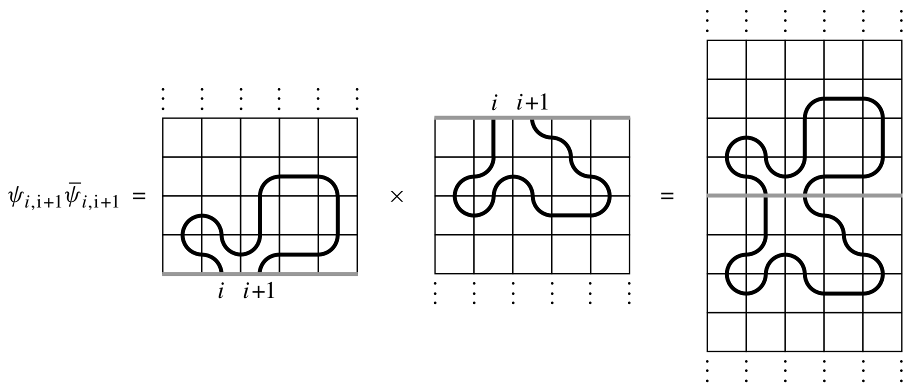

Figure 4. This is an example of a contribution of a single term in (1). Here, the connectivity pattern α and β are simple links joining two neighbouring points at positions i and $i+1$. Periodic boundary conditions are implicit, the two vertical boundaries of each strip must be identified; the grey line corresponds to the free boundary at $y = 0$ and the dots mean that the cylinders continue to infinity.

It is possible to show that Z_L is a solution of an inhomogeneous recurrence relation of the form[1]

$$Z_L = AZ_{L-1} + F, \tag{2}$$

where A is a known function and $F = \psi_{i,i+1}\bar{\psi}_{i,i+1}$, where $\psi_{i,i+1}$ ($\bar{\psi}_{i,i+1}$) is the (dual) component which corresponds to the connectivity pattern that has the neighbouring sites i and $i+1$ joined (see Figure 4). Using the relation of the loop model to the IK vertex model and the vertex operator realization of the algebra underlying the IK model [3] we can write a multiple integral formula for the components $\psi_{i,i+1}$ and $\bar{\psi}_{i,i+1}$. Therefore, formula (2) provides us with another approach to study polymer chains in two dimensions.

References

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[1] This formula holds for the inhomogeneous model in which we have additionally L parameters $z_1, \ldots, z_L$. Under certain specialization of z_i and z_{i+1} for $i = 1, \ldots, L$ one finds (2). We omitted the index i in (2) for simplicity.