# Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 44. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, Faculty of Science and Technology, Federation University Australia, PO Box 663, Ballarat, Victoria 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 44 is 15 November 2015. The solutions to Puzzle Corner 43 will appear in a future issue of the *Gazette*.

*Notice:* If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

### Triangular territory

Consider a finite set of points in the plane. Suppose that the area of the triangle formed by any three points is at most 1. Prove that the entire set of points must lie in a triangle whose area is at most 4.

### Perpendicular cuts

Let an *irregular pizza* be a region in the plane which is closed, bounded and has a well-defined area. Prove that every irregular pizza can be cut into four pieces of equal area using two straight and mutually perpendicular cuts.

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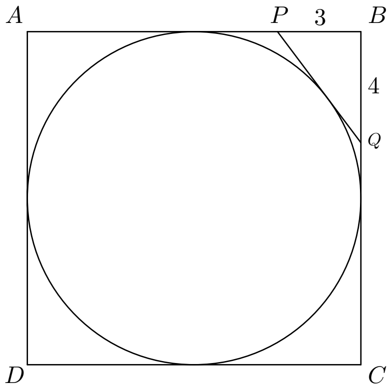
## Inscribed radius

Let $ABCD$ be a square with an inscribed circle. Let $P$ and $Q$ be points on sides $AB$ and $BC$, respectively, such that $PQ$ is tangent to the circle. If $PB = 3$ and $QB = 4$, what is the radius of the circle?




## Friendly division

Any two people are either friends or not friends. Given a group of people, is it always possible to divide them into two groups such that for any person, at least half of his/her friends are in the opposite group?

## Squaring off

(i) Amy and Bob are playing a game on an unmarked $n \times n$ chessboard. Amy begins by marking a corner square. Then Bob marks an unmarked square which is adjacent to (sharing an edge with) the square Amy just marked. Then Amy marks an unmarked square which is adjacent to the square Bob just marked. Then it is Bob's turn again and so on. This process continues until one of them can no longer make a valid move and loses the game. Who has a winning strategy?

(ii) If Amy's first move is to mark a square adjacent to a corner square, who has the winning strategy?

## Solutions to Puzzle Corner 42

Many thanks to everyone who submitted. The \$50 book voucher for the best submission to Puzzle Corner 42 is awarded to Kevin McAvaney. Congratulations!

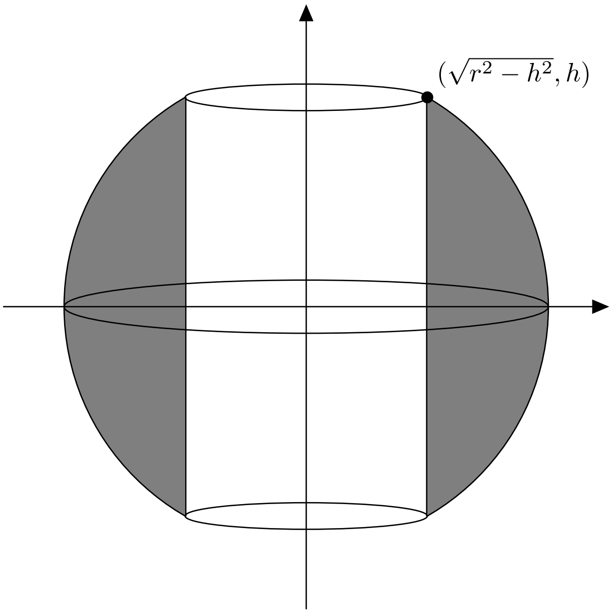
### Volume valuation

*A spherical ball has a cylindrical hole drilled through its centre. Prove that the remaining volume only depends on the length of the cylindrical hole.*

*Solution by Steve Clarke:* Let the sphere have radius $r$ and let the length of the cylindrical hole be $2h$. Then the radius of the hole is $\sqrt{r^2-h^2}$. The remaining volume can now be computed using the following integral:

$$2\int_{\sqrt{r^2-h^2}}^{r} 2\pi x\sqrt{r^2-x^2}\,\mathrm{d}x = -\frac{4\pi}{3}\big[(r^2-x^2)^{3/2}\big]_{\sqrt{r^2-h^2}}^{r} = \frac{4\pi h^3}{3}.$$

Thus the remaining volume only depends on $h$.




### Random subsets

*Let $S$ be a set with $n$ elements. Sammy randomly chooses a subset of $S$. Sally also randomly chooses a subset of $S$. What is the probability of Sammy's set being a subset of Sally's set?*

*Solution by Jensen Lai:* Instead of choosing a random subset of $S$, it is equivalent to independently choose each element with a probability of $1/2$. Let us fix any particular element $a \in S$. There are four equally likely possibilities regarding who chooses $a$:

both, neither, only Sally, only Sammy.

Call the element $a$ *good* if one of the first three possibilities occurs. It is clear that in order for Sammy's set to be a subset of Sally's set, $a$ must be good. The probability of $a$ being good is $3/4$.

Since the required subset condition holds if and only if every element is good, and the elements are chosen independently, the required probability must be $(3/4)^n$.

### Musical musing

*Six musicians are attending a music festival. At each scheduled concert, some of them may perform while the others listen as members of the audience. How many such concerts are needed so that every musician has a chance to listen, as a member of the audience, to every other musician?*

*Solution by Kevin McAvaney:* Four concerts are needed to fulfil the requirements. A schedule of the concerts must include all possible ordered pairs $(a, p)$ of musicians, where $a$ is the audience while $p$ is playing. There are $6 \times 5 = 30$ such ordered pairs.

In any concert, if exactly $m$ musicians are playing, the number of ordered pairs covered is $m(6-m)$. This is maximised if $m = 3$ and $m(6-m) = 9$. Hence three concerts can cover at most $3 \times 9 = 27$ ordered pairs. This is insufficient since we require 30 ordered pairs.

To show that four concerts are sufficient, number the musicians 1 to 6 and use the following construction.

| Performers | 456 | 235 | 136 | 124 |
|---|---|---|---|---|
| Audience | 123 | 146 | 245 | 356 |

It is easy to check that every ordered pair is covered by this construction.

### Repeated rummage

*There are $n+1$ cards, each having a number between 1 and $n$. You know that every number between 1 and $n$ appears exactly once, except for one number which appears twice. The cards are placed in a row, face down on the table. Furthermore you know that they are sorted in ascending order from left to right. How many cards do you need to turn over in order to determine the repeating number?*

*Solution by Dave Johnson:* Let $f(n)$ be the minimum number of flips for $n+1$ cards. Note that the leftmost card is always 1 and the rightmost card is always $n$, whereas the $i$th card from the left $(1 < i < n)$ can be either $i-1$ or $i$. We will prove a few facts about $f(n)$.

First of all, $f(n)$ is non-decreasing, or $f(n) \leq f(n+1)$. To show this, suppose there are only $n+1$ cards to begin with. We can simply add an additional card labelled $n+1$ on the right, then apply the algorithm for $n+2$ cards to find the repeating number in $f(n+1)$ flips. Therefore $f(n)$, the number of flips required for $n+1$ cards, is at most $f(n+1)$.

Next, $f(2^k) \leq k$. Consider the case where there are $2^k+1$ cards. Let us first flip over the middle card. There are two cases. If the middle card is $2^{k-1}$, then the repeating number must be in the range $[1, 2^{k-1}]$. If the middle card is $2^{k-1}+1$, then the repeating number must be in the range $[2^{k-1}+1, 2^k]$. In both cases, the problem is now equivalent to determining the minimum number of flips required for $2^{k-1}+1$ cards (including the middle card), or $f(2^{k-1})$. Repeating this argument $k-1$ times, we are left with only 3 cards, which can be resolved using only 1 flip. Therefore $f(2^k) \leq k$ is proven.

Finally, $f(2^k+1) \geq k+1$. For the sake of contradiction, suppose that only $k$ flips are needed for $2^k+2$ cards. In particular, the repeating number can take $2^k+1$ possible values. But since each flip has at most two possible outcomes, we cannot distinguish between more than $2^k$ scenarios. This is a contradiction.

Combining everything, i.e. $f(n)$ is non-decreasing, $f(2^k) \leq k$ and $f(2^k+1) \geq k+1$, it is clear that the unique function satisfying all the requirements is $f(n) = \lceil \log_2 n \rceil$. Therefore the minimum number of flips required to identify the repeating number is $\lceil \log_2 n \rceil$.

### Suitable suitor

*A king is choosing a bridegroom for his daughter. There are three suitors available, a knight, a knave and a commoner. The king knows that the knight always tells the truth, the knave always lies and the commoner can do either. The king would like to avoid choosing the commoner, but he does not know who is who.*

(i) *Suppose the three men do not know each other. If the king can ask each man a yes/no question, what should he ask to find a suitable bridegroom?*

(ii) *Suppose the three men know each other. If the king can only ask one man a single yes/no question, what should he ask to find a suitable bridegroom?*

*Solution by Aaron Hassan:* (i) The king can ask each suitor a question which is always true (e.g. 'Does $1+1=2$ hold?'). The knight will answer 'yes', the knave will answer 'no'. The commoner will agree with exactly one of them. To choose the bridegroom, the king can simply choose the person who answered differently to the other two. This method will always avoid the commoner.

(ii) The king can arrange the three suitors in a horizontal line. For convenience let us label them $A$, $B$ and $C$ from left to right. He could then ask $A$ the following question: 'From my perspective, is the knave standing directly (i.e. adjacent) to the left of the commoner?' If the answer is 'yes', then $B$ should be chosen. If the answer is 'no', then $C$ should be chosen.

To show that this method works as intended, let us consider the possible cases.

- If $A$ is the knight, then the answer is truthful. 'Yes' implies $B$ is the knave, while 'no' implies $B$ is the commoner. The knave is chosen as the bridegroom.
- If $A$ is the knave, then the answer is a lie. 'Yes' implies $B$ is the knight, while 'no' implies $B$ is the commoner. The knight is chosen as the bridegroom.
- If $A$ is the commoner, then the commoner is always avoided since $A$ is not chosen regardless of the reply.

Therefore this method always avoids choosing the commoner as the bridegroom.

Ivan is a Postdoctoral Research Associate in the School of Mathematics and Applied Statistics at The University of Wollongong. His research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.