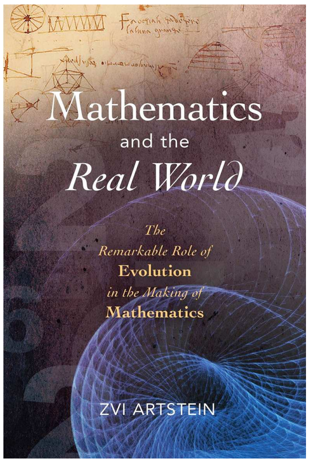
# Book Reviews

## Mathematics and the Real World

Zvi Artstein
Prometheus Books, 2014, ISBN 978-1-61614-091-5

Zvi Artstein is a distinguished Professor of Mathematics, specialising in Control Theory and Game Theory, at The Weizmann Institute of Science in Israel. This book, subtitled 'The remarkable role of Evolution in the making of Mathematics', is a translation from the Hebrew, so we may assume that its original target audience was lay persons and mathematics students and teachers in Israel. Its intent is to describe the relationship between mathematics and the physical and social world, based on the concept that intellectual advances are driven by evolutionary pressure.




The book is a historical and philosophical account of the development of mathematics with emphasis on aspects such as simple arithmetic, spatial visualisation and pattern recognition which can be related to evolution by natural selection. This is an interesting approach, well worth serious consideration. Unfortunately, the author adduces no evidence from evolutionary anthropology, biological genetics, DNA analysis or cognitive science to support his theories, relying rather on folk psychology and 'Just–So' stories. Unsurprisingly, he deduces that most applied mathematics but precious little pure mathematics has an evolutionary advantage.

Interspersed in the text, which generally avoids technicalities, are paragraphs in a different font, which the author invites the uninterested reader to skip, and which are supposed to contain more sophisticated mathematical arguments. Unfortunately, these sections themselves are often historically inaccurate and mathematically misleading. Here for example in its entirety, is Artstein's explanation of how Newton verified Galileo's observation on the path of a falling object:

> Newton showed that the derivative of the function $a(t) = \alpha t^n$, where $\alpha$ is a constant, is $\alpha n t^{n-1}$. In particular, if the second derivative has a fixed value $g$, its integral is $gt$, and the integral of the latter is $\frac{1}{2}gt^2$. This shows that as the Earth's gravitational pull $g$ is constant over short distances, the parabolas that Galileo observed when he dropped bodies from the top of a tower fulfilled Newton's second law of motion.

The historical account, ranging from tallying by pre-historic humans, astronomy and mensuration in bronze-age agricultural societies, Greek mathematics and medieval astronomy, through Galileo and Newton to the early moderns such as Euler and the Bernoullis, contains nothing novel and is riddled with errors, of which the most egregious are: the Babylonians inscribed calculations on potsherds; Euclid developed axiomatics in order to avoid optical illusions in diagrams; Greek mathematicians studied the brachistochrone problem; Fermat's Last Theorem occurs among Hilbert's list of unsolved problems presented at the 1900 International Mathematical Congress. The author is on firmer ground when dealing with the history of the mathematics with which he is most familiar: the principle of least action, calculus of variations, statistics and the social sciences and computation. He also has a perceptive chapter on the nature of research in mathematics.

The book contains frequent observations on how mathematics should be taught and learned. However, the author's remarks on mathematics education are, in the reviewer's opinion, one-sided and extreme.

Some of the foregoing negative comments must have filtered through to the author or his publisher during the process of publication, because Artstein concludes with a poignant Afterword, imploring readers to forgive his errors, since Evolution did not prepare us for error-free rigorous analysis!

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## Neverending Fractions
## An introduction to Continued Fractions*

Jonathan Borwein, Alf van der Poorten, Jeffrey Shallit and Wadim Zudilin
Australian Mathematical Society Lecture Series, number 23
Cambridge University Press, 2014, ISBN 978-1107-00665-2
Also available in eBook (ISBN 978-0-511-90265-9)

Continued fractions form a classical area within number theory, the roots of which can be traced back to Euclid's algorithm for the greatest common divisor of two integers (300 BC). Several centuries ago, Rafael Bombelli (1579), Pietro Cataldi (1613), and John Wallis (1695) developed the method of continued fractions for rational approximations of irrational numbers (such as square roots), and later on great mathematicians like Leonhard Euler (1737 and 1748), Johann Lambert (1761), Joseph L. Lagrange (1768 and 1770), Carl Friedrich Gauss (1813), and others discovered various fundamental properties and important applications of

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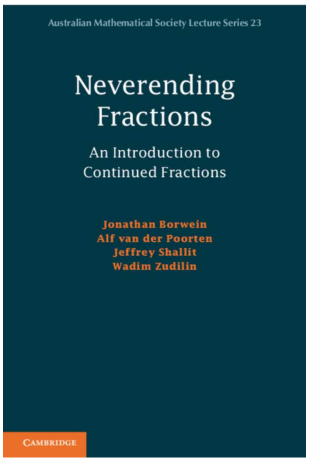
continued fractions. In fact, these fascinating objects have been a very active field of research ever since, and the vast contemporary literature on continued fractions evidently shows that this topic is still far from being exhausted.

The book under review grew out of many lectures that the four authors delivered independently on different occasions to students of different levels. Its main goal is to provide an introduction to continued fractions for a wide audience of readers, including graduate students, postgraduates, researchers as well as teachers and even amateurs in mathematics. As the authors point out in the preface, their intention is to demonstrate that continued fractions represent a neverending research field, with a wealth of results elementary enough to be explained to this target readership.

Regarding the precise contents, the book comprises nine chapters, each of which is divided into several sections. While the first three chapters are devoted to a general introduction to continued fractions, the subsequent six chapters deal with more special topics and applications of the theory.




Chapter 1 presents the necessary prerequisites from elementary number theory with full proofs. These concern the following themes: divisibility of integers and the Euclidean algorithm, prime numbers and the fundamental theorem of arithmetic, Fibonacci numbers and the complexity of the Euclidean algorithm, approximation of real numbers by rationals and Farey sequences. Chapter 2 begins the study of continued fractions and their algebraic theory, thereby explaining the continued fraction of a real number in general, the principle of Diophantine approximation, the continued fraction of a quadratic irrational and the Euler–Lagrange theorem in this context, the construction of real numbers with bounded partial quotients, and other results on rational approximation.

Chapter 3 touches upon the metric theory of continued fractions, with emphasis on the growth of partial quotients of a continued fraction of a real number, the approximation of almost all real numbers by rationals, and the classical Gauss–Kuzmin statistics in metric number theory. Chapters 4, 5 and 6 originate in lectures that one of the authors, the late Alf van der Poorten (1942–2010), gave in the last few years before his untimely death.

Chapter 4 is titled 'Quadratic irrationals through a magnifier' and contains some informal lectures on continued fractions of algebraic numbers, Pell's equation, and some concrete examples.

Chapter 5 is a survey of aspects of continued fractions in function fields, with a view toward some so-called (recursively defined) Somos sequences, pseudo-elliptic integrals, and hyperelliptic curves, whereas Chapter 6 briefly discusses the relationship between neverending paper foldings and continued fractions. Chapter 7 provides the study of a class of generating functions that are connected to remarkable continued fractions and rational approximations. Lambert series expansions of generating functions and an inhomogeneous Diophantine approximation algorithm are the main tools applied here.

Chapter 8 treats the Erdős–Moser equation

$$1^k + 2^k + \cdots + (m-2)^k + (m-1)^k = m^k$$

and its possible integer solutions for $m \geq 2$ and $k \geq 2$.

A conjecture by P. Erdős states that such solutions do not exist, and L. Moser proved in 1953 that only for even exponents $k$ and rather large integers $m$ such solutions could be expected at all.

In this chapter, both the arithmetic and the analysis of the Erdős–Moser equation are outlined, where efficient ways of computing certain associated continued fractions as well as explicit bounds for solutions are presented. The basic reference for this chapter is the recent paper by Y. Gallot, P. Moree and W. Zudilin (*Math. Comput.* **80,** No. 274, 1221–1237 (2011; Zbl 1231.11038)).

The concluding Chapter 9 finally turns to irregular continued fractions by surveying their general theory as well as some important examples, including Gauss' irregular continued fraction for the hypergeometric function, Ramanujan's arithmetic-geometric mean (AGM) continued fraction (from his second notebook) and related developments by one of the authors of the present book (J. Borwein) and his collaborators, an irregular continued fraction for the zeta value $\zeta(2) = \pi^2/6$, and a new proof of R. Apéry's theorem on the irrationality of $\zeta(3)$ as a striking application of the foregoing discussion.

There is an appendix to the main text containing a collection of interesting continued fractions, both regular and irregular, where most of those represent special real numbers, values of special functions, particular infinite series, and some $q$-series, respectively. As one can see, the book is a combination of formal and informal styles of expository writing, and a mixture of introductory textbook and topical surveys likewise. Many of the special topics discussed in the later chapters are not to be found in other books but only in scattered articles and lectures. As for full details with regard to these topics chapters, the reader is referred to the original research papers listed in the rich bibliography. In fact, each chapter ends with a set of notes providing additional remarks and hints for further reading, and a few exercises invite the reader to acquire complementary knowledge through independent work.

All together, the present book gives a beautiful panoramic view of the 'neverending story of neverending fractions' by making apparent their naturalness, their ubiquity, and their wide-range of applications in very lucid and inspiring a manner.

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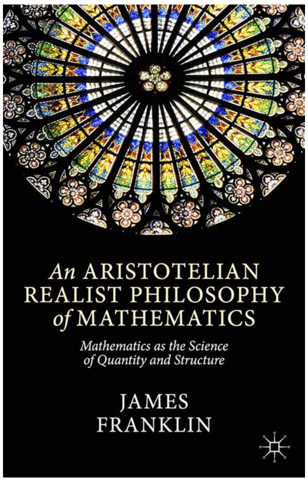
## An Aristotelian Realist Philosophy of Mathematics

James Franklin
Palgrave Macmillan, 2014, ISBN 978-1-137-40072-7

What are the current trends in philosophy of mathematics, and what relevance do they have to the practice of mathematics? Those are some of the questions I asked myself when I began reading this book by James Franklin, Professor of Mathematics at UNSW and founder of the ‘Sydney School’ in the philosophy of mathematics.




There are a few hurdles to overcome for a lay person attempting to read a book or paper in philosophy. The first is the jargon. Some phrases, such as ‘epistemology and ontology’, ‘mereology’, ‘potentially infinite’ (which just means unbounded) are clearly defined in dictionaries and Wikipedia and pose no real problem. But others, such as ‘necessary’, ‘contingent’, and ‘uninstantiated universal’ are ambiguous or depend on individual psychology. Even more troubling are words such as ‘random’ which have different meanings in mathematics and philosophy. A striking example is the claim by the author that the sequence of decimal digits of $\pi$ and even a finite initial segment of this sequence, is random. Random, it seems, is in the eye of the beholder.

A second feature of the philosophical literature is its polemical tone. Unlike the polite euphemisms one finds in mathematical papers which point out errors or incomplete proofs, philosophical papers bristle with words like ‘mistaken’, ‘ill-informed’ and ‘falsity’. In mathematics I have only seen something approaching this in papers on the foundation of probability espousing the frequentist or Bayesian approach.

So what are the principal opposing schools in mathematical philosophy today? According to Franklin they are Platonism, which holds that mathematics is about the real world, and Nominalism which claims it is about words. They correspond roughly to what mathematicians call platonism and formalism. (I use lower case for the mathematical concepts to distinguish them from the slightly different philosophical meaning). But whereas mathematicians see no contradiction in embracing platonism when doing research and formalism when writing it up, philosophers view this as akin to treason.

Within Platonism there are again two competing views of mathematics. One is Platonic Idealism, which holds that the real world is an approximation to mathematics, and the other is Aristotelian Realism which holds that mathematics is an approximation to the real world. The author is firmly in the latter school, proclaiming mathematics to be the science of quantity and structure. For example, the number 6 is the property of a heap of six apples which distinguishes it from a heap of five apples. Similarly, 1 meter is the length of a chalk line measured with a standard rod, and a sphere is the shape of a bronze ball. Franklin is well aware that a straight line drawn on paper is neither straight nor a line, a 1000-sided regular polygon is indistinguishable from a circle and a bronze ball is not a sphere. He deals with this problem by associating with each real entity a tolerance, in the engineering sense.

For Aristotelian Realists, the space in which we live is locally Euclidean and its properties are not postulated, but observed and verified. The arithmetic of the integers and their ratios make sense, and we can define an action of the rationals on the continuum. Since there is no natural choice of a unit of measure, there is no embedding of the rationals in the real line, so it is not obvious if and how the arithmetic of the real and complex numbers is related to that of the integers and rationals. The topology of space and hence continuous real functions can be defined intrinsically, but Franklin does not address the question of smoothness and the existence of pathological functions.

The claim that mathematics concerns the real world extends also to structures. Franklin defines a mathematical property to be structural if it can be defined in terms of the concepts 'same and different' and 'part and whole'. For example an entity is symmetrical if it consists of two parts which are the same in some respect. What 'in some respect' means is apparently a property of human cognition. Among the problems with this concept are the cognitive errors of apophenia (the human tendency to perceive patterns in random or meaningless information), and the opposite error of failing to recognise a pattern that exists. Terry Tao, for example, sees relations between the eigenvalues of the random matrix and the distribution of zeroes of the Riemann zeta function that I do not.

A major claim of the author is that while pure mathematicians may be more comfortable in the Platonic universe, Aristotelian Realism is especially appropriate for applied mathematics. One breathtaking suggestion is that the applied mathematician need not be concerned with infinity because applied mathematics only deals with finite objects. This concentration on small concrete cases has the unfortunate consequence that the author fails to take account of the importance of generalisation. For example, according to Franklin, the purpose of Euler's Königsberg Bridges paper was to prove that the burgers of Königsberg could not stroll over all their bridges just once.

A major problem associated with Realist mathematics is the question of infinity, both discrete and continuous. The author recognises that it must be faced in any coherent mathematical system, and his partial solution is the notion of 'uninstantiated universals', that is, entities that could exist but do not, the philosopher's favourite example being the golden mountain. By invoking uninstantiated universals, Franklin allows Realist mathematics to admit large cardinals when required. But when explaining the reliance of mathematics on logical reasoning, he fails to address the problem of the internal consistency of Aristotelian Realism. There should be no need for this, because if mathematics is only about the real world, it has a model. But as soon as uninstantiated universals enter the picture, so does the possibility of logical contradiction. For example using arguments valid in Realism with uninstantiated universals, Gödel constructed a consistent universe in which $2^{\aleph_0} = \aleph_1$, and Cohen one in which $2^{\aleph_0} > \aleph_1$. Whereas formalist mathematicians are happy to pursue the consequences of accepting or denying the continuum hypothesis, to the Realist their truth or falsity is a property of the real world.

It is easy to ridicule such attempts to rewrite mathematics. It is more useful to consider which aspects of mathematics can be developed using an Aristotelian Realistic foundation when we allow a cautious interpretation of uninstantiated universals. Firstly ZF Set Theory, including the axiom of infinity but without the unrestricted axiom of choice, is realistic and can be used to define functions and relations. Furthermore, the arithmetic of the integers and their ratios make sense. We can embed the field of rationals in a complete Archimedean ordered field and so construct real and complex numbers and vector spaces with their associated topology. However, notions of compactness that require choice are not allowed, so that classical analytic and harmonic analysis remain out of reach. Elementary number theory makes sense and so does finite combinatorics. We can define algebraic structures, but lacking the maximum principle, we cannot prove the existence or properties of maximal normal substructures except in the finite case.

To my mind, too much is lost without a platonic and formalistic approach.

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