Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 42. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, Faculty of Science and Technology, Federation University Australia, PO Box 663, Ballarat, Victoria 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 42 is 15 July 2015. The solutions to Puzzle Corner 42 will appear in Puzzle Corner 44 in the September 2015 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Volume valuation

A spherical ball has a cylindrical hole drilled through its centre. Prove that the remaining volume only depends on the length of the cylindrical hole.

Random subsets

Let S be a set with n elements. Sammy randomly chooses a subset of S. Sally also randomly chooses a subset of S. What is the probability of Sammy's set being a subset of Sally's set?

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Musical musing

Six musicians are attending a music festival. At each scheduled concert, some of them may perform while the others listen as members of the audience. How many such concerts are needed so that every musician has a chance to listen, as a member of the audience, to every other musician?

Repeated rummage

There are $n + 1$ cards, each having a number between 1 and n. You know that every number between 1 and n appears exactly once, except for one number which appears twice. The cards are placed in a row, face down on the table. Furthermore you know that they are sorted in ascending order from left to right. How many cards do you need to turn over in order to determine the repeating number?

Suitable suitor

A king is choosing a bridegroom for his daughter. There are three suitors available, a knight, a knave and a commoner. The king knows that the knight always tells the truth, the knave always lies and the commoner can do either. The king would like to avoid choosing the commoner, but he does not know who is who.

(i) Suppose the three men do not know each other. If the king can ask each man a yes/no question, what should he ask to find a suitable bridegroom?

(ii) Suppose the three men know each other. If the king can only ask one man a single yes/no question, what should he ask to find a suitable bridegroom?

Solutions to Puzzle Corner 40

Many thanks to everyone who submitted. The \$50 book voucher for the best submission to Puzzle Corner 40 is awarded to Jensen Lai. Congratulations!

Rolling riddle

On average, how many times do you have to roll a die before all six numbers appear at least once?

Solution by Steve Clarke: The answer is 14.7. If $k < 6$ of the numbers have already appeared, the chance of rolling a new number is $\frac{6-k}{6}$. Using the standard result for geometric distributions, the expected number of attempts before we get a new number is $\frac{6}{6-k}$.

Therefore the expected number of rolls to obtain all 6 numbers is given by

$$\frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 14.7.$$

Balanced views

Given a convex polygon $A_1A_2\cdots A_n$ in the plane, we say a point P (in the same plane) is balanced *if*

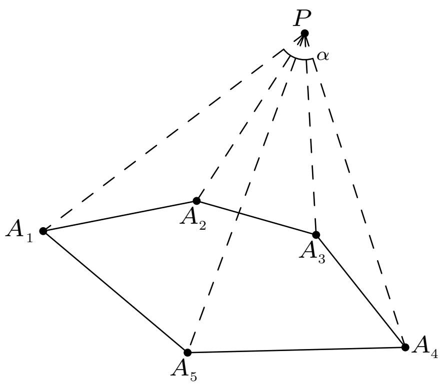
$$\angle A_1PA_2 = \angle A_2PA_3 = \cdots = \angle A_{n-1}PA_n = \angle A_nPA_1.$$

(i) *Prove that for any convex polygon with an odd number of sides, there is at most one balanced point in the plane.*

(ii) *Can there ever be more than one balanced point if the convex polygon has an even number of sides?*

Solution by Jensen Lai: (i) A balanced point P cannot coincide with a vertex of the polygon, as then some of the angles will be undefined. Furthermore, a balanced point P cannot lie on an edge of the polygon, otherwise all relevant angles have to be $180°$ and the convex polygon degenerates into a straight line. So we have ruled out the possibility of P lying on the perimeter of the polygon.

Begin with a convex polygon $A_1A_2\cdots A_n$ where n is odd. Suppose there exists a balanced point P outside of the polygon. Consider the convex hull of $A_1, A_2, \ldots, A_n$ and P. Let the angle of the convex hull at P be α, as shown in the following diagram.



By the definition of balanced points, we must have

$$\theta = \angle A_1PA_2 = \angle A_2PA_3 = \cdots = \angle A_{n-1}PA_n = \angle A_nPA_1. \tag{1}$$

The angle θ cannot be $0°$, otherwise all points must be collinear and the polygon is degenerate. Since P is outside of the original polygon, some of the angles in (1) are oriented clockwise while others are oriented anticlockwise. Let the number of clockwise angles be p and the number of anticlockwise angles be q, where $p + q = n$. If we sum up the clockwise and the anticlockwise angles separately from each other, we must have

$$\alpha = p\theta = q\theta \quad \Longrightarrow \quad p = q.$$

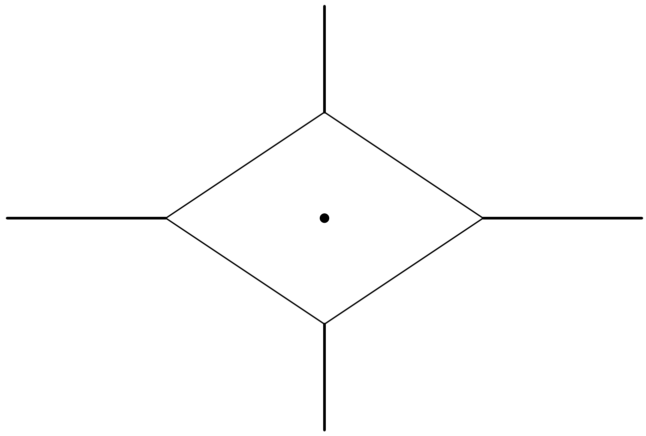
This is a contradiction since $n = p + q$ is odd.

It suffices to show that there is at most one balanced point inside the polygon. For any balanced point P, all angles in (1) must be oriented in the same direction (e.g. clockwise). Hence $\theta = \frac{360°}{n}$. If there is a second balanced point Q distinct from P, we must have

$$\angle A_1PA_2 = \angle A_1QA_2 = \frac{360°}{n}.$$

This implies that A_1PQA_2 is a cyclic quadrilateral. By the same argument A_2PQA_3 is also a cyclic quadrilateral. Since a circle is defined by three points, all five points A_1, A_2, A_3, P and Q must all lie on the same circle. Repeating the argument for $A_4, A_5, \ldots$, we see that the points $A_1, A_2, \ldots, A_n, P$ and Q all lie on a single circle. This is a contradiction since P and Q lie inside the convex polygon $A_1A_2 \cdots A_n$. Therefore there can be at most one balanced point.

(ii) In the case of n being even, the proof in part (i) also shows that there can be at most one internal balanced point. However, there may be multiple external balanced points. For example, in the case of a rhombus, the centre as well as all points on the extensions of the diagonals are balanced points.



Spherical stroll

An ant is crawling on the surface of a sphere whose radius is one metre. After a while, the ant returns to its starting position. Prove that if the ant has crawled no more than 2π metres, then its path can be contained in some hemisphere of the sphere.

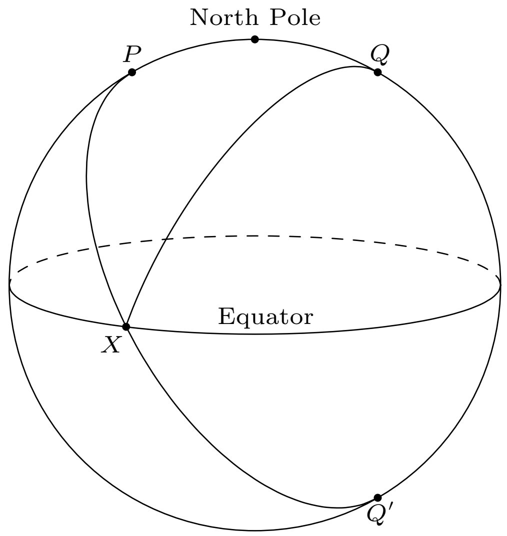
Solution: For any two points A and B on the surface of the sphere, denote by AB the shortest path between A and B along the surface. If A and B are antipodal points, there are multiple shortest paths, each in the shape of a semicircle. In this case, the usage of AB will avoided.

It suffices to assume that the ant has crawled exactly 2π metres, as it can always take additional round trips to make up the distance. Let the starting point be P and let the 'halfway point' (when the ant has crawled exactly π metres) be Q. If P and Q are antipodal points, then the shortest distance between P and Q is exactly π metres. This means that the ant's journey consists of two semicircles between P and Q. It is clear that we can choose a hemisphere which contains both semicircles.

Now suppose P and Q are not antipodal points. Without loss of generality, let the midpoint of PQ be the north pole. We shall prove that the ant's journey is contained in the northern hemisphere.

For the sake of contradiction, assume that the ant leaves the northern hemisphere during the journey $P \to Q$. This means it must cross the equator at some point, say X. In particular, the ant travels a total of π metres in the journey $P \to X \to Q$. Let us reflect the second half of this journey, $X \to Q$, about the equator to obtain the journey $X \to Q'$. Thus Q' is the reflection of Q and the journey $P \to X \to Q'$ is also π metres long.

Recall that the midpoint of PQ is the north pole, this implies that P and Q' are antipodal points. Since the shortest distance between P and Q' is π metres, the journey $P \to X \to Q'$ must be a semicircular path consists of PX and XQ'. This means that the original journey $P \to X \to Q$ must consist of the shortest paths PX and XQ. But both PX and XQ are still contained in the north hemisphere. This is a contradiction and the solution is complete.



Digital division

Consider the set of all five-digit numbers whose decimal representation is a permutation of digits 1, 2, 3, 4 and 5. Is it possible to divide this set into two groups, so that the sum of the squares of the numbers in each group is the same?

Solution by Joe Kupka: Yes it is possible. Consider the permutations which start with 12 and divide them into two sets:

$$G = \{12345, 12453, 12534\}, \qquad H = \{12354, 12543, 12435\}.$$

It is clear that $\sum_{n\in G} n = \sum_{n\in H} n$. Using this fact, we have the following sequence of equalities:

$$\begin{aligned}\sum_{n\in G} n^2 - (66666-n)^2 &= \sum_{n\in G} 66666(2n-66666)\\ &= \sum_{n\in H} 66666(2n-66666)\\ &= \sum_{n\in H} n^2 - (66666-n)^2\end{aligned}$$

Rearranging gives us

$$\sum_{n\in G} n^2 + \sum_{n\in H}(66666-n)^2 = \sum_{n\in H} n^2 + \sum_{n\in G}(66666-n)^2. \tag{2}$$

When n is a five-digit permutation starting with 12, the number $66666-n$ is a five-digit permutation starting with 54. Thus equation (2) has taken all five-digit permutations starting with either 12 or 54, and divided them into two sets with equal sums of squares.

We can repeat this argument for any two starting digits. Denote by S_{ab} the set of the numbers with ab as the first two digits. There are 20 such sets, each with 6 numbers. Since the digit strings ab and $(6-a)(6-b)$ are always distinct, we may pair up S_{ab} with $S_{(6-a)(6-b)}$ to from 10 groups of 12 numbers. Using (2), each group can be divided into two sets with equal sums of squares. Combining them gives the desired result.

Note: As evident from the solution, there is a lot of flexibility in how the numbers can be divided. Another neat idea is to divide them based on their permutation parity.

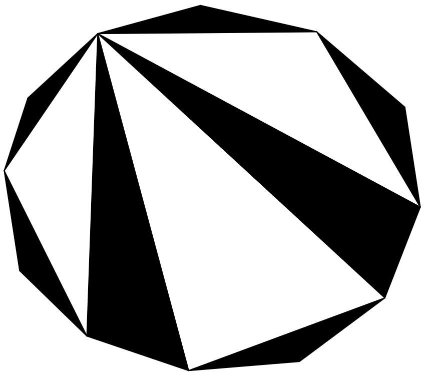
Tricky triangulation

For $n \geq 3$, a convex n-gon can be divided into $n-2$ triangles by using $n-3$ of its diagonals. This is called a triangulation*. For which values of n is it possible to triangulate a convex n-gon such that every vertex is adjacent to an odd number of the resulting triangles?*

Solution by Jensen Lai: Such a triangulation is possible if and only if n is a multiple of 3.

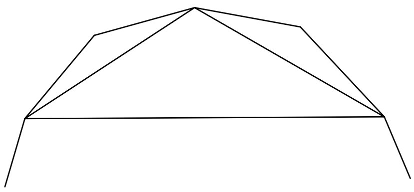
Consider an n-gon with a valid triangulation. Colour the triangles in the following way. First, select a triangle touching the perimeter of the n-gon and colour it black. Then, use white to colour all triangles sharing an edge with the black triangle. Then, use black again to colour all triangles sharing an edge with any of these white triangles. Continue to colour the triangles in this fashion until all $n-2$ triangles have been coloured.

Since all triangles are formed by diagonals and sides of the n-gon, there can be no conflicts in colouring as there is only one path to each triangle from the original black triangle. Furthermore, we note that no two triangles of the same colour share an edge and every diagonal borders a black triangle and a white triangle.



Since each vertex of the n-gon is adjacent to an odd number of triangles, the two triangles touching the perimeter of the n-gon must have the same colour. Hence, just like the starting black triangle, all triangles touching the perimeter must be black. In other words, every white triangle is formed by three diagonals. If the number of white triangles is w, there must be $3w$ diagonals. Since there are exactly $n - 3$ diagonals, n must be a multiple of 3.

It remains to construct a valid triangulation whenever n is a multiple of 3. This can be done inductively. The base case of $n = 3$ is trivial. Given an existing valid n-gon triangulation, a valid $(n + 3)$-gon triangulation can be formed by attaching the following structure to an existing side.



Ivan is a Postdoctoral Research Fellow in the School of Mathematics and Applied Statistics at The University of Wollongong. His current research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.