# Book Reviews

**Origami$^5$**

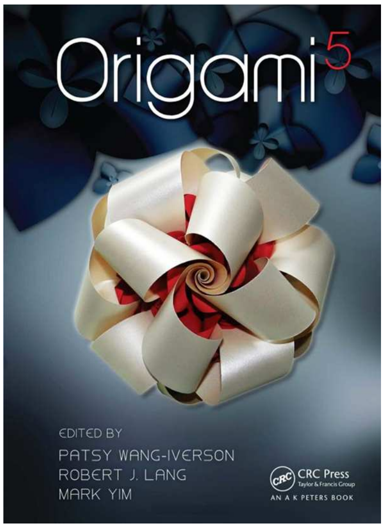
Patsy Wang-Iverson, Robert J. Lang and Mark Yim, Editors
CRC Press, Boca Raton, FL, 2011, ISBN 978-1568817149

## Introduction

This fine volume — in my opinion a cornucopia of good things — is the proceedings of the 'Fifth International Meeting of Origami Science, Mathematics, and Education' although as opposed to most conference proceedings, much of this material is accessible to the general reader. There is in fact a fourth section (which is first in the volume), not listed in the subtitle, on 'Origami History, Art, and Design'.

There are too many delights in this volume to discuss each one at length; so I will discuss some sample chapters from each section, and finish with some general remarks.

## 1. Origami History, Art, and Design




The first chapter, 'History of Origami in the East and West before Interfusion', by Koshiro Hatori, starts off by vigorously refuting the common belief that origami originated in China. There appears to be no evidence to support this; however, origami developed independently in both Japan and the West. Japanese folds used paper with different shapes, as well as cuts, and the results were often painted. Western folding was stricter, using mostly square or rectangular paper, and no cuts. Japanese folds, often as wrappers for gifts, were embedded in the samurai tradition; Western folding had its antecedents in the folding of baptismal certificates. What we now understand as 'origami' is a remarkable fusion of both the Japanese and Western forms.

A paper with the interesting title 'Simulation of Nonzero Gaussian Curvature in Origami by Curved-Crease Couplets', is not fully of heady mathematics, but more of a description of how curved surfaces can be generated by a sort of pleating technique. A 'curved-crease couplet' consists of two consecutive folds, one a mountain (or ridge); the other a valley, where one fold is straight and the other curved. Many pictures illustrate the richness of this approach to producing three dimensional curved figures; there is a brief digression on ruled surfaces.

Another paper looks at the extraordinary curved-crease sculptures of David Huffman, who you might know as being the inventor of Huffman codes. This section finishes off with a paper about 'oribotics' (a fusion of origami and robotics), and discusses a sort of folded flower which opens and closes depending on the amount of light falling on it. Some of this chapter is devoted to discussing the material which is used here: as well as paper, the creator has used polyester which is 'cooked' in a steam oven to stabilize its folds.

## 2. Origami in Education

Origami as an educational tool has a long history in Japan, and in primary schools in the west. There seems to be a growing interest in using origami as an educational tool in secondary and post-secondary education; one article has the enticing title 'Origami and Spatial Thinking in College-Age Students', and reports on the use of origami as part of general studies program at an American tertiary college. The article, however, spends most of the time discussing Likert scales, the use of ANCOVA to measure the different scores between groups of students, and is short on particulars: what the students actually *did*, and what they were expected to learn. The author's conclusion was that students did indeed increase their spatial reasoning skills, but also warns carefully that not all of this increase is necessarily attributable to the use of origami.

An article 'My Favorite Origamics Lessons on the Volume of Solids' (here 'origamics' means 'the mathematics of origami') pulls together origami, Cauchy's mean value theorem, and various elementary optimization problems — the sort of 'word problems' which have been trotted out for ever to first-year students. It may well be that some ideas in this article could be used to great effect to aid three-dimensional thinking. I don't know about *you*, but many of *my* students have a great deal of difficulty moving between the description of an object and its abstract representation.

However, my favorite article in this section is 'Narratives of Success: Teaching Origami in Low-Income Urban Communities', which reports on the use of origami in a Chicago school, the student body of which was predominantly African-American and Hispanic, most students having poor literacy and numeracy skills as well as learning difficulties. 'For the girls, that often meant they had been pregnant . . . , and/or had spent time in jail. For the boys, it usually meant they had been in jail and/or had an unstable home life or no home at all.' The author describes helping the children first make origami models and then write stories about them, and how the combination led to real learning from the first time. Note that the students here had a true poverty of learning: the idea that paper could be used to fold shapes and animals or flowers was completely new to them: '. . . I did home visits where the only visible thing in the house was a crack pipe or a stained mattress'. Anybody who doubts the transformative power of education would do well to read this article.

## 3. Origami Science, Engineering, and Technology

You would expect that origami would have many applications in the sciences and engineering, which indeed it does, and engineers are beginning to sit up and take notice. Thus in 'The Origami Crash Box' the authors explore, using amongst other tools finite element methods, how to create a crumple zone, such as in car, which would absorb most of the force in a crash while leaving the occupants unhurt. Such modelling and construction is now fundamental to car manufacture, and this article investigates how origami modelling techniques can be used in the design of such crash boxes.

Another article: 'Origami Folding: A Structural Engineering Approach' looks at the structural properties of sheets folded into a textured pattern. The bending properties, stiffness, rigidity and strength of such sheets is shown to have many applications, from lighter and stronger cardboard, to sheets which must undergo deformations, such as the skin of aircraft wings. This article is mostly discursive, with only minimal mathematics, but I don't think that detracts from its interest.

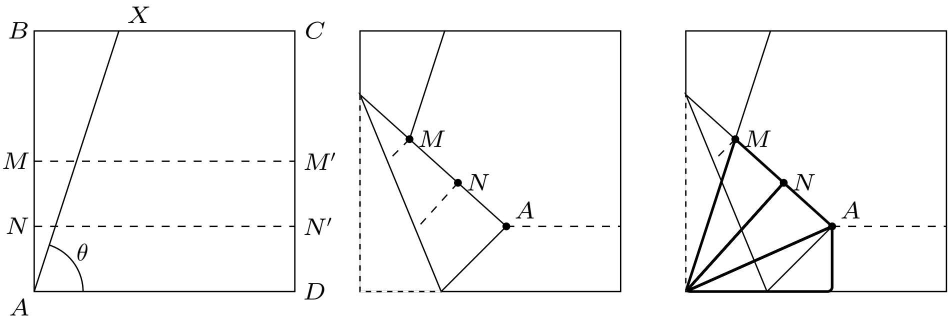
## 4. Mathematics of Origami

Here is where it gets most interesting, at least for the readers of *this* review. But before I launch into a discussion of the articles, some background. Much of the mathematics of origami has in antecedents in a set of axioms developed by the French mathematician Jacques Justin (who seems to have been the first to enumerate them in full), the Italian-Japanese mathematician Humiaki Huzita, and more recently still by Koshiro Hatori. The axioms are referred to by any non-empty subset of Huzita–Hatori–Justin. These axioms enumerate precisely the folds which are possible, and the first few are pretty much what you'd expect: a point can be folded onto another point, two points can be joined by a fold which passes through them both, a crease can be folded onto another crease, and so on. Note that the axioms only concern straight folds. With these axioms it is possible to construct all points constructible by (unmarked) ruler and (collapsible) compass. However, there is one axiom which describes a construction which has no Euclidean equivalent:

> Given two points $p_1$ and $p_2$, and two lines $L_1$ and $L_2$, there is a fold which places $p_1$ on $L_1$ and $p_2$ on $L_2$.

It can be easily shown that the construction of this fold is algebraically equivalent to solving a quartic equation; hence with origami constructions it is possible to trisect any angle, or construct irrational cube roots.

For example, Figure 1 shows one construction for trisecting an angle.




(a) Initial folds

(b) Fold to find points of trisection

(c) Proof that the construction works

Figure 1: Trisecting an angle

Start with a square $ABCD$; the angle $XAD$ is to be trisected. Crease the paper in half along $MM'$ and fold the bottom edge $AD$ to $MM'$ and out again, making a crease one-quarter up the paper through $NN'$. Now fold the bottom left corner in such a way that point $M$ lands on line $AX$ and $A$ lands on line $NN'$. Then the lines from the new positions of $A$ and $N$ to the bottom left will trisect the angle. The right-most diagram shows why this construction works; all the triangles are the same.

A clever construction of $\sqrt[3]{2}$ is similar and is shown in Figure 2: start with a square creased into vertical thirds. Fold the bottom left corner to the top edge in such a way that the left vertical crease lies on the right vertical crease.

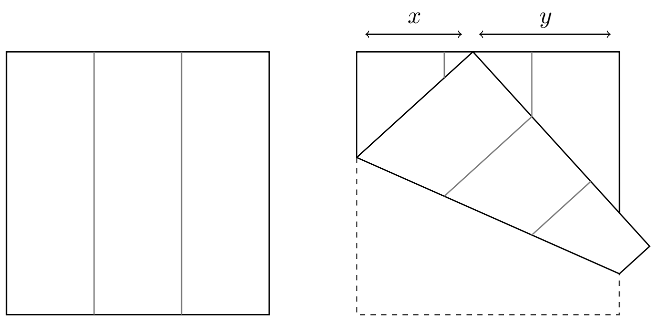



Figure 2: Origami cube root

Then the position of the corner along the top edge divides that edge into a ratio of $y/x = \sqrt[3]{2}$.

These folds are equivalent to the Greek *neusis* construction, which uses a marked ruler: that is, a ruler with two marks on it; and with which it is also possible to trisect angles and construct irrational cube roots.

There are many different lines of origami research; one explores the concept of a 'multifold' where several folds are made simultaneously so that various edges and points line up. Although a double fold is just possible with fiddling, higher degree multifolds would seem to severely stretch the bounds of practical folding. However, by using multifolds it can be shown that higher degree polynomial equations are solvable. Another direction of generalizing is multidimensional origami, where instead of folding a plane along a line, one folds a 3-space along a plane. This must of course remain purely theoretical, and yet you'd expect some fascinating geometry.

This section is the longest. The first article: 'An Introduction to Tape Knots' is by Jun Maekawa, who is known for some fundamental theorems about crease patterns. Tape knots are obtained by knotting up strips of paper: that a pentagon can be made from an overhand knot is well known. The article first explores polygons of odd and even sides, and then looks at the structures of the knots themselves, with some crossing diagrams. It would be fascinating to know if the polygonal nature of the final knot has any relation with some of the standard knot invariants. The paper also mentions links, along with some crossing diagrams.

Erik Demaine from MIT is the coauthor of several papers, one of which is a small, neat paper 'Folding Any Orthogonal Maze' which describes precisely that: an algorithm for folding a maze out of paper, using what Demaine calls 'gadgets'; for example parts of the maze corresponding to a single wall, a corner where two walls meet. Although there is no proof, the paper makes the claim that the algorithm is optimal in the sense of requiring the smallest grid for the maze.

Roger Alperin, well known for his research into the mathematics of origami, is represented here by 'Origami Alignments and Constructions in the Hyperbolic Plane', using for the purpose the Cayley–Klein model, where the plane is a disk, and lines on the plane correspond to lines on the disk. The article introduces six basic constructions, and shows that these can be used to construct all points in the hyperbolic plane which could be constructed with compass and marked ruler. As mentioned above, in the Euclidean plane such constructions are called *neusis* constructions; it is known that classical origami and neusis constructions are equivalent in the sense of producing the same set of constructible points. Alperin shows here that this result is also true in the hyperbolic plane.

This section ends with what is, to me, a very elegant high point: 'Circle Packing for Origami Design is Hard', by Erik Demaine, Sándor Fekete, and Robert Lang. This paper considers the problem of packing circles into a square: such packings are the first step in generating a 'crease pattern' for the creation of a model with lots of points, such as an insect. The centres of the circles will become the points such as legs, antennae, wings. This paper proves that finding an optimal packing is NP-hard. Such problems form a superset of NP-complete problems, and may be considered to be at least as hard as any of them. The authors briefly discuss approaches to minimizing the size of paper required to contain a given set of circles; this research is ongoing.

## Final remarks

Most of the mathematics in the book, especially in the first three sections, tends to be discursive and general; this is to be expected of a conference which is only in part mathematical. You should not be misled by this into thinking that the mathematics of origami is trivial or shallow; origami research is increasingly calling on more and more branches of mathematics, with published articles referencing Galois Theory, combinatorial, algebraic and differential geometry, Gröbner bases and much else.

This volume is thus a fascinating snapshot into the current world of origami research, and of the many applications in which the theory and practice of origami are leading to new insights.

A minor complaint is that the relatively low printing resolution has meant that the fine detail in some diagrams — and most photos — is obscured or blurred. In such a diagram-rich discipline as origami, precision of diagrams is not just a luxury but a necessity. To some extent this is ameliorated by a handsome inclusion of colour plates, but even so I wish that some diagrams had been re-drawn for greater clarity.

A very nice touch is the inclusion of an index.

I think every academic library should have a copy — at least electronically. Whether you are a research mathematician, a mathematics educator, whether you are a professional or an interested amateur, whether you are in a research hiatus and looking for a project, or whether you are looking for a project for your students, you could do much worse than investigate this volume. You may find material which sparks an idea. And even if not, you will have been exposed to some elegant and delightful mathematics.

Alasdair McAndrew
College of Engineering and Science, Victoria University, PO Box 14428, Melbourne, VIC 8001.
Email address: Alasdair.McAndrew@vu.edu.au

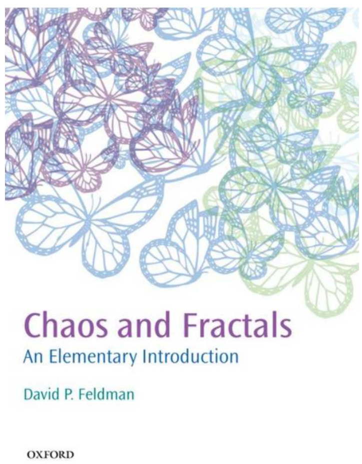
◇ ◇ ◇ ◇ ◇ ◇

## Chaos and Fractals: An Elementary Introduction

David P. Feldman
Oxford University Press, 2012, ISBN 978-0-19-956643-3

They say not to judge a book by its cover, but if one can't do that, then one can judge it by the format of its content. On that basis this book obtains top marks. Each page has only one column of text that is about two-thirds the width of the page, so there is good deal of white space. In that column of white space the author has on some occasions footnotes, or captions for a figure that is in the main text column. Also at the start of each chapter is the section headings for the chapter, with their pages numbers, listed in that one-third column. So the presentation of this book is one of clarity and an invitation to come read. An important aspect for any book, but certainly more so for a text book, which is essentially what this book is. I was impressed with its layout and presentation, which of course puts the reader in a positive frame of mind when tackling the guts of the matter, that is the text. So top marks from me on that aspect.

Also there are exercises at the end of each chapter, but unfortunately no answers, not even for selected exercises, which was somewhat disappointing. Some of the exercises mirror closely the worked examples in the body of the text and some lead onto other concepts in the next chapter. All round a thoughtful approach, despite there being no answers. However there is more. At the end of some chapters there is a section titled 'Further Reading', with the first appearance being in chapter zero and the next not until chapter seven. Still another nice touch that is eminently helpful both for student and casual reader like myself.




The book is divided into seven sections that are in order: 'Introduction to Discrete Dynamical Systems', 'Chaos', 'Fractals', 'Julia Sets and the Mandelbrot Set', 'Higher-Dimensional Systems', a 'Conclusion' along with the ubiquitous 'Appendices'. As can be seen the core of the matter is sections two through to five. The first section, 'Introduction to Discrete Dynamical Systems', is the only odd one out in that its material is pretty basic and most chapters would have been familiar to, or at least taught to a student in the first years of high school. For example the first chapter covers different ways of viewing a function; as a formula; as a graph; as a map and so forth. The second chapter covers iterating a function. So all of this would bore a tertiary student pretty quickly, which is a pity because the core material is a treat. Interestingly, in the preface the author notes that students have reported the slow start to the book due to the eight chapters of section one being in the way of the core material. It would be easy to include seven of those chapters in the Appendices as revision material for the student who needed to do that. Otherwise I fear he would lose some students before they arrive at the meat of the book.

The sections from ‘Chaos’ to ‘Higher-Dimensional Systems’ are well set out, clear and amply illustrated with graphs that are worth a thousand words. Mr Feldman covers aperiodic behaviour; the sensitivity of initial conditions; the bifurcation diagram plots histograms of chaotic orbits; chaotic systems being sources of randomness; the dimensions of fractals using the favourites such as snowflakes, Cantor sets, Sierpinski triangle; as well as random fractals. So as can been seen from that small sample in just the areas of Fractals and Chaos, all the major areas are covered, along with suggested reading if one wishes to extend their knowledge. Similarly for the Julia Sets, Mandelbrot Sets, Discrete Dynamical Systems, Lorenz Attractors and One-dimensional Cellular Automata, there are clear explanations of them all along with many graphs and exercises to keep the student or intrigued recreational mathematician enthralled. There is much more besides the brief examples I have given above. All that I can do is give you an indication of the breadth of material in this book. The rest is up to you.

The book isn’t without some typos but they are reasonably obvious and don’t detract from the overall value and impact of the book. A nice surprise was the mention on a couple occasions in the book of Michael Barnsley’s work with Fractals. I enjoyed an interesting Fractal talk of Michael’s at a recent EViMS Conference at ANU. It was a good prelude to reviewing this book.

There is an Appendix chapter devoted to further reading that covers a selection of books from a popular style to more advanced texts, as well as a few online resources. These are of course a very brief selection and if your thirst has been whetted then there is an impressive References section to troll through. All this is rounded out with a short index. So for any lecturer or teacher looking for a text on these subjects, this book is worthy of your consideration. For the student or layperson interested in these subjects it is a good read and could be read without completing the exercises. However, you would miss out on half the fun, these subjects just begged to be played with. If you haven’t guessed already I recommend this book.

Gordon Clarke
Email address: gordonc53@gmail.com

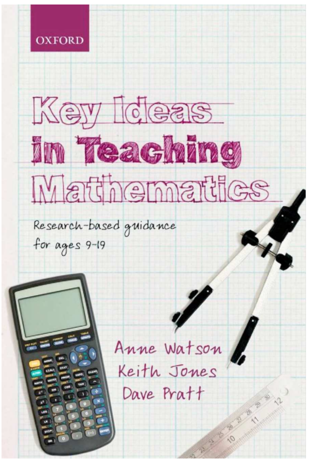
◇ ◇ ◇ ◇ ◇ ◇

## Key Ideas in Teaching Mathematics: Research-based guidance for ages 9–19

Anne Watson, Keith Jones and Dave Pratt
Oxford University Press, 2013, ISBN 978-0199665518

### A few opening remarks

Most academic mathematicians are involved in some teaching, and up until recently few, if any, had any formal educational qualifications. This means that most tertiary mathematics educators have little grounding in educational theory, or of current research and practice in mathematics education. Such teachers tend to teach either the way they were taught themselves, or a method they have evolved which works for them. An immediate result is the preponderance of 'chalk and talk' in lectures, and 'drill and practice' in tutorials. As well there are constant complaints about the poor standard of students 'nowadays': they can't do algebra, they don't know how logarithms work, they don't seem to be able to understand how a function and its graph are related. Come on, own up — when was the last time you complained about the ill-preparedness of your first year students?




It seems to me that anybody who teaches mathematics at any level owes it to themselves and their students to have a basic working knowledge of educational theory, and in particular as it relates to the teaching of mathematics. Tertiary teachers should also have some understanding of school curricula and school teaching: what are secondary students being taught, and how are they learning it?

This book is timely, and has the enormous advantage of being written clearly and simply. The authors have eschewed jargon for clarity — a very sensible decision which I wish more education authors would emulate. In Australia, as elsewhere, tertiary mathematics educators are struggling with heavy workloads, competing demands of research and administration, and an increased internationalization of the student cohort. This makes it all the more necessary that such teachers look at how and why students learn the way they do, and learn why students have their difficulties.

Although this book may appear, from its title, to have little interest for the practising tertiary mathematics educator, nothing could be further from the truth. This book should be essential reading for all.

## The book and its contents

This book grew out of a study funded by the Nuffield Foundation in 2008 into how children aged 6–16 learn mathematics. This book is a synthesis of that research, and embodies a learner-centred paradigm: "We have viewed mathematics as developing 'bottom-up' through learning rather than solely 'top-down' from an academic viewpoint." There are seven 'domains' about which the book is structured, with one chapter for each: relations between quantities and algebraic expressions; ratio and proportional reasoning; connecting measurement and decimals; spatial and geometrical reasoning; reasoning about data; reasoning about uncertainty; functional relations between variables. A final chapter looks at moving beyond basic mathematics to more advanced material such as may be developed further at a post-secondary level.

The book contains numerous QR codes by which the reader can immediately be taken to an online resource at the nuffieldfoundation.org site. This is quite a good idea; it's hard to think of an easier way of moving between printed and online material.

In the first 'domain' chapter, 'Relations between quantities and algebraic expressions', there is a long discussion about the many ways in which notation is confusing: for students in early years, the concept of using a 'letter' instead of a 'number' can give rise to many difficulties. The list of misconceptions of algebraic notation include:

- treat letters as shorthand, for example $a$ = apple;
- some students believe that different letters have to have different values, so would not accept $x = y = 1$ as a solution to $3x + 5y = 8$;
- different symbolic rules apply in algebra and arithmetic, for example '2 lots of $x$' is written '$2x$' but 'two lots of 7' are not written '27'.

These misconceptions, if left unnoticed or not managed in a timely fashion, can seriously impact a student's learning of more mathematics. The authors describe some of the research into the teaching and learning of algebra and note that there is no 'best sequence' for teaching algebra. They approvingly note the usefulness of computer algebra systems (mentioning Mathematica, Maple), and with regard to drill and practice, quote a researcher who comments on the notion that this is always a Good Thing: 'if the students spend enough time practicing dull, meaningless, incomprehensible little rituals... something WONDERFUL will happen'. This is not to say that drill and practice doesn't have its place, but the exercises must be carefully chosen so that the notions become fully internalized ('such as happens with reading, writing, learning dance steps, and so on'), in which case 'something wonderful *can* happen'. This chapter also includes various possible approaches for teaching algebra, along with some comments about the advantages and limitations of each approach.

The next chapter, on ratio and proportional reasoning, points out that these concepts are extremely difficult, mainly because of the many ways they are used. The concept 'fraction' can be simply defined in a number of ways, but as is so often the case a formal definition is no help for elementary teaching. It may seem that fractions are trivial, but even in post-secondary classes we find students who seem to have difficulties with the conceptual handling of fractions — let alone the algebra associated with them! The authors claim that students need to be exposed early and often to the many uses and meanings associated with fractions and ratios: 'This is the strongest recommendation to emerge from this chapter: the need to provide students with repeated and varied experiences, over time, so that multiple occurrences of the words and the associated ideas and methods can be met, used, and connected.' The authors note that this topic is hard, partly because much of it depends on dealing with the equality of two fractions: $a/b = c/d$; the fractions perhaps being expressed as ratios. Students have to decide what to multiply, or divide, and by how much, and in 'mixing problems' (mixing orange juice and water, for example), they can't reduce the problem to counting.

'Connecting measurement and decimals' sounds like kindergarten material — when was the last time you talked seriously about 'decimals'? And yet even supposedly well-educated adults have difficulty here. Recent research has uncovered some extraordinary gaps in understanding, such as people believing that, for example, 0.13 was bigger than 0.7 because '13 is bigger than 7'. Apparently simple notions such as place value can be daunting, and require careful and precise teaching. Measurement is inextricably linked with counting, and here we have other difficulties such as students attempting to find the area of a rectangle by adding the length of its sides. The authors call for a stronger link between measurement and decimals at the secondary school level.

'Spatial and geometric reasoning' points out that such reasoning can be powerful because of its intuitive nature, and quote Sir Michael Atiyah saying just that. The philosopher Jean Piaget is quoted as pointing out that whereas academic geometry moves from measurement to shape analysis then along to topology, a child's intuition goes the other way: topological (how many holes?) through to identifying shapes, and finally to measurement. A major problem as noted here — and still unsolved — is the vexed issue of how much geometry to include in any curriculum, and where it should go? The panoply of new computational tools ('Dynamic Geometry Software') such as Cabri Geometry, Geometers Sketchpad, GeoGebra, Cinderella don't so much as solve this problem as bring problems about curriculum design and sequencing into 'sharp relief'. The authors note that research into the use of such tools for learning geometry is relatively new, but seem to consider that such tools can have — and indeed, should have — a part in any modern curriculum.

The next chapter, 'Reasoning about data', looks at statistics, and wonders if mathematics educators should be teaching statistics at all, or whether it is in fact a separate discipline. Statistics certainly spreads across the curriculum, but simply for practicality it seems unlikely that in a school to be taught by anybody other than a mathematics teacher. The chapter notes some research gaps: for example, how do children conceive of a 'sample'? The authors point out that graphing tools (such as TinkerPlots) are invaluable in allowing students to reason about data 'without an impossible threshold of calculation and graphing to overcome'. And indeed, one of the pleasing aspects of the book in general is the authors' implicit — and at times explicit — approval of technology as a valid and valuable teaching tool. The following chapter, 'Reasoning about uncertainty' follows on naturally from statistics, and again the use of graphical tools is noted. There is a whistle-stop tour through some research about the understanding of chance, of randomness, and on the use of simulations with a software tool. Much about probability learning is under-researched, including student reasoning about situations which are only partly determined, as well as risk-analysis, or rather risk-based decision making. It is clear that our knowledge of how children of any age — and learners in general — learn about probability, randomness and risk is surprisingly sketchy. And this should be no surprise, given the general poor understanding across the population.

The final two chapters, on functions, and on moving beyond school, look at matters which, for many tertiary educators, are major issues: the difficulty many students have of moving between functional, graphical, and tabular representations of a function; and also of different representations of a function. For example, a quadratic function may be equally represented as $y = ax^2+bx+c$, $y = a(x-b)^2+c$, $y = a(x-b)(x-c)$ and yet students may well be mystified by a representation different to the one they already know. They may know that for $y = ax^2+bx+c$ the value $c$ is the $y$-intercept, but what is the $y$-intercept in $y = a(x-b)^2+c$? For that matter, what *do* the values $a$, $b$ and $c$ represent? And as with most areas of mathematics, there is no research which indicates that there is a 'best' sequence for teaching about functions. The final chapter briefly discusses trigonometry, and difficulties encountered in moving from $\sin(x)$ as a triangle measurement to a real-valued, and then complex-valued function; calculus and analysis—differentiation, and $\epsilon$-$\delta$ proofs of continuity and differentiability; and finally formal statistical inference. At these levels there is very little research, and the authors frankly admit that much is unknown about how these students learn, and what practices may best support their learning.

## A few final remarks

This book is not about how to teach mathematics, although it contains plenty of good ideas, especially about sequencing of topics. It is a synthesis of recent research about how children of various ages learn specific mathematical topics, and about how much is still unknown.

There are several strands running through the book, of which two of the strongest (so it seemed to me) were the importance of context ('what's this stuff used for?') and the promise of technology.

It is no secret, given the general innumeracy of much of the population, that mathematics teaching has a long way to go. I think that this neat, accessible book should be widely read: by mathematics teachers, by mathematicians, by educational policy makers. The more we understand how children (and older students) learn mathematics, and the sorts of misunderstandings that impede their progress, the better we will be able to build a curriculum which engages, enthuses, and excites its audience.

Alasdair McAndrew
College of Engineering and Science, Victoria University, PO Box 14428, Melbourne, VIC 8001.
Email address: Alasdair.McAndrew@vu.edu.au

◇ ◇ ◇ ◇ ◇ ◇

## The Fascinating World Of Graph Theory

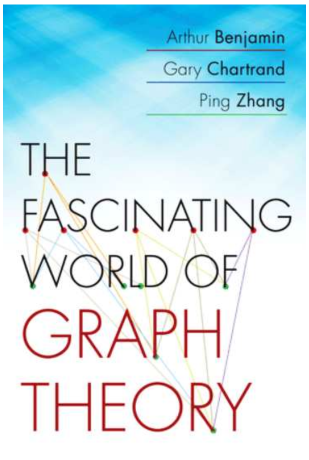
Arthur Benjamin, Gary Chartrand and Ping Zhang
Princeton University Press, 2015, ISBN 978-0-691-16381-9

Over the years many people have asked me about the usefulness of mathematics in some endeavour that interests them. What I have discovered is that the ensuing conversation is largely driven by the poser's mathematical background. People who have taken no mathematics past high school or only a course or two at the tertiary level do not understand what mathematics brings to the table. Many times I have begun an explanation of an approach to a problem and the typical response is 'That's mathematics?' Therein lies one of the problems facing mathematics and mathematicians, namely, the vast majority of people do not know what mathematics is. Unfortunately, many of those people are in influential positions in society.

An interesting question, given the above comments, is what should we offer in the way of a course to a college student who is going to take a single mathematics course? This review is not going to make any attempt to discuss such a highly loaded question. However, I have raised the issue because the book under review has been written from the viewpoint of what graph theory is about and the kinds of contexts in which graph theory may be used as a model for a realistic problem. It could well serve as the kind of book we would give to someone wanting to learn something about the spirit of mathematics.

Chapter one deals exclusively with games, puzzles and problems that may be modelled using graphs. The models are introduced along with a basic description of what graphs and multigraphs are, and how they are used to capture the situations under consideration. This material would be understandable to a curious intelligent person with only a basic mathematical background. Such a person might even think to herself/himself that much of the discussion deals with problems they might not have considered as mathematics prior to reading the book.

The second chapter introduces the notion of classifying graphs in some way. The basic idea of isomorphism is introduced, pushing the boundaries of thinking about degree, and the first discussion of an unsolved problem arises. This is done via the reconstruction problem. Introducing the reader to the fact that there are problems which have not yielded to vast research efforts likely is an eye-opener for many of them. I have met a fair number of people who wonder what one can possibly research in mathematics.




Chapter three introduces basic notions revolving around connectivity and distance. Both vertex and edge cuts are discussed along with several interesting applications. In addition, there is a charming discussion of both Erdös numbers and pseudonyms several groups have used for publications. Another characteristic of the book continues to emerge here. There are five theorems stated in this chapter and three of them are given without proof.

Chapter four introduces trees and basic properties of these useful objects. Cayley's formula for labelled trees is treated thoroughly as are minimal spanning trees. I would have liked to have seen a discussion of Steiner trees for two reasons. First, there are nice examples of their usefulness compared to minimal spanning trees. Two, the huge gap in difficulty between trying to find a minimal Steiner tree and a minimal spanning tree is a nice illustration of how a small tweaking of what one is looking for in an application may profoundly change the problem.

Chapters five and six deal with graph traversals. There is a nice discussion of both Euler tours and the Chinese Postman Problem, both of which are edge traversals, in the first of the two chapters. The book is richly infused with history and background. One item which particularly delighted me, and relates to my comments at the beginning, was the following excerpt from a letter written by Euler to the mayor of Danzig after the former had solved the Königsberg bridges problem.

> Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others.

Vertex traversals are covered in Chapter six. These are, of course, Hamilton cycles and there is a nice discussion of the history behind this topic. The chapter concludes with a discussion of the Travelling Salesman Problem which is a famous and important optimization problem involving Hamilton cycles. It is in some sense the vertex analogue of the Chinese Postman Problem although the latter allows traversal of an edge more than once, whereas, the Travelling Salesman Problem allows passage through each vertex precisely once.

Chapters seven and eight deal with graph decompositions, that is, partitions of the edge set of a graph. Chapter seven restricts itself mostly to matchings, perfect matchings and decompositions into perfect matchings. There is a brief excursion into decompositions into 2-factors. The subsequent chapter deals with other decomposition problems with an emphasis on cycle decompositions. The chapter concludes with a detailed analysis of the puzzle known by many names, but probably the best known contemporary name is Instant Insanity.

Chapter nine begins with an interesting history of Herbert Robbins whose only paper in graph theory dealt with orienting a graph so that the resulting digraph is strongly connected. This chapter deals with orientations of graphs with an emphasis on tournaments, that is, orientations of complete graphs. The chapter concludes with a nice application of tournaments for voting schemes.

The material in Chapter 10 is well-presented standard material on topological graph theory. However, I confess that I was disappointed because when I first saw only the title of the chapter, I was pleased to see the inclusion of material on graph drawing. I say this because the topic almost is never discussed in a book, and I know of several companies whose primary business is essentially producing nice drawings of graphs. So the thought that a reader was going to be exposed to some of the subtle ideas in trying to convey information to the public, board members, employees, etc. via nice drawings of various relational structures was appealing.

The last two chapters deal with colouring. Chapter 11 looks at vertex colouring while Chapter 12 deals with edge colouring. Consistent with the rest of the book, there is an interweaving of history, motivation, applications and basic results.

The book is rich with history and really brings life to the people who have developed the subject of graph theory. The authors have done a splendid job of showing the reader that thinking about and doing mathematics is a human endeavour. Anyone reading this book will take that away for certain. That brings us to the ultimate question: Who is going to find this book useful?

The book would be attractive to an intelligent reader who knows some mathematics and would like to get insight into graph theory. The level is such that a rich background in mathematics is not necessary. There are proofs but almost all are easy to follow.

I don't believe this book would work well for the standard presentation of a course on graph theory. I do believe it would work well for a group of students who are prepared to read, think about what they read, be prepared to discuss what they are reading, and willing to explore items arising from their discussions. Most of us do not have the luxury of teaching courses in such a manner. There is a very good set of exercises helping in developing ideas arising in the book.

Brian Alspach
School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308.
Email address: brian.alspach@newcastle.edu.au

◇ ◇ ◇ ◇ ◇ ◇

## Lozanovsky's Notebooks

Grigorii Yakovlevich Lozanovsky

Translated from the Russian by Marek Wójtowicz

Part I. Edited by Włodzimierz Odyniec, Aleksandr I. Veksler and Marek Wójtowicz. Pedagogical University of Zielona Góra Press, Zielona Góra, 2000. ISBN: 83-7268-031-0

Part II. Edited by Marek Wójtowicz. Kazimierz Wielki University Press, Bydgoszcz, 2012. ISBN: 978-83-7096-855-7

Part III. Edited by Marek Wójtowicz. Kazimierz Wielki University Press, Bydgoszcz, 2012. ISBN: 978-83-7096-862-5

This is a very interesting and unusual piece of mathematical writing. The only immediate analogue which comes to mind is a comparison with the famed 'Scottish book' used by members of the Lvov Mathematical School to jot down various mathematical problems, and which determined later an early development of Banach space geometry. The books under review are an English translation from the Russian original due to an outstanding Russian mathematician, Grigorii Lozanovsky[1]. From the outset, I must emphasize a host of difficulties which were facing the translators and editors, primarily Marek Wójtowicz, who undertook this mammoth task. Indeed, the project began only in 1998, long after the untimely death of the author in 1976. The three volumes of the book consist of 2223 Problems/Questions/Remarks which are frequently supplied with many comments by A.I. Veksler, L. Maligranda, M. Mastyło, W. Odyniec, W. Wnuk, D. Yost and the scientific editors.

Lozanovskii was a leader in the St. Petersburg school of Banach lattices and semi-ordered spaces and his life and work has left an indelible trace on his friends and associates. I did not meet Lozanovskii in person, but I had heard about him from many former colleagues from St. Petersburg (and also from my native Tashkent) who have always referred to him as 'an outstanding mathematician and human being'. The text of his notebooks confirms the incredible power of Lozanovskii's insight concerning interesting and important questions about (special classes of) Banach lattices and Function Spaces, Interpolation Theory and various parts of Banach space theory, and a host of other topics which he came across. The text also is a testament to his enormous passion for Mathematics. The text was not intended for publication and is unstructured; the work done by the editors and translators is simply outstanding. He has made an important contribution to Interpolation Theory, in particular to the complex method of interpolation. I will especially emphasize two outstanding contributions made by Lozanovskii. One of the most important constructions in Interpolation Theory is now named the Calderón-Lozanovskii construction (see e.g. [18]) to properly credit his contribution to that area. A reader can observe through Problems 636, 973, 987, 1000,

[1]Lozanovskii would be a more suitable transliteration of the surname.

1364, 1467, 1472, 1605, 1780, 1861 how thoroughly Lozanovskii developed that construction and studied its properties. This construction retains its importance until now and practically anyone who would be interpolating couples of Banach lattices (or their noncommutative counterparts) would be using (some form of) Lozanovskii's results. From my personal experience, I can cite a recent paper [5] in which questions related to noncommutative integration (relevant to the noncommutative geometry of Alain Connes [8]) were treated with substantial use of the ideas underlying the Calderón–Lozanovskii construction. Another outstanding result by Lozanovskii is his factorization theorem stated in Problems 628 and 629 in the second volume of the Notebooks. This theorem asserts that the pointwise product of any Banach function space $E$ with the Fatou property and its Köthe dual $E'$ is an $L_1$-space. For a far reaching generalization of Lozanovskii's factorization theorem I refer to a very recent paper [13] (I thank L. Maligranda for this reference).

Of course, some of the problems stated (or, frequently, somewhat vaguely suggested) in his notebooks have lost their shine, and it is not surprising, after all the publication has come almost 40 years too late .... However, even now I can assure experts (and simply people who have an interest in these parts of Mathematics) that their reading would be a richly rewarding experience. I will try to convince a reader by my own example: in this review I shall identify several topics from the notebooks, which are very close to my own research (and heart) and which were (with incredible insight) identified and 'predicted' by Lozanovskii. However, prior to referring to my own experience, let me state unequivocally that Lozanovskii had an incredible ability to unmistakably identify central problems in Banach lattices and allied areas of Banach space geometry. Just one example: Problems 409 and 463 are directly relevant to the famous unconditional basic sequence problem resolved by Gowers and Maurey [10] in 1993. More examples of that kind can be found, however, it is easier for this reviewer to concentrate on those problems in the book with which he is well familiar and which at the same time have not been discussed in the comments of subsequent commentators. My list is intended to convince the potential readers of the incredibly wide spectrum of Lozanovskii's interests and deep interconnections which he surmised.

Problem 132 asks whether the triangle inequality $|A+B| \le |A|+|B|$ holds for two Hermitian and non-commuting operators on a Hilbert space. This question has been thoroughly investigated in [22] and answered there in the negative. However, a positive answer can be given if we replace the classical order on the set of all Hermitian (bounded) operators with the (so-called) submajorization (or else, Hardy–Littlewood–Pólya) preordering (see also [17]). The latter inequality plays a useful role in various questions concerning the geometry of symmetric operator spaces, which are a noncommutative extension of classical rearrangement invariant function spaces (see [7], [17] and the references therein).

Problems 654 and 1123 are concerned with Lozanovskii's observation that a combination of (local) measure convergence and weak convergence in Lebesgue spaces $L^1$ on $\sigma$-finite measure spaces yields norm convergence. In Problem 654, he suggested to thoroughly examine this property. It should be pointed out that this property is strongly linked with the classical characterization due to Dunford and Pettis of relatively weakly compact sets in $L_1$-spaces. In particular, a bounded subset $A$ of any abstract $L$-space is relatively weakly compact if and only if each disjoint sequence in its solid hull converges in norm to zero. In turn, each of these statements is equivalent to the assertion that $A$ is of uniformly absolutely continuous norm. This leads naturally to a study of spaces with the property that norm convergence of sequences is equivalent to weak convergence plus convergence for the measure topology. The study of such spaces seems to have been initiated in [14] and [15], where the term $(wm)$-property (for rearrangement-invariant spaces with such a property) was coined. The study of such spaces has been on-going for the last two decades and the author of the present review is in a position to give an update.

The analogue of the Dunford–Pettis criteria for the classical Lorentz spaces was obtained in [23] and later, in [6, Corollary 1.4] it was established that every Lorentz space $\Lambda_\phi$ has the $(wm)$-property. Orlicz spaces on the interval $[0, 1)$ with property $(wm)$ have been fully characterized in [1]. Finally, in [9, Proposition 6.10] it is shown that, in rearrangement-invariant function spaces on measure spaces with finite measure possessing the property $(wm)$, each relatively weakly compact subset is of uniformly absolutely continuous norm. The latter result does not hold when the measure space is equipped with an infinite measure. Furthermore, the just cited results hold also in a much greater generality when rearrangement-invariant function spaces are replaced with their noncommutative counterparts [9].

Problem 1520 asks whether every separable symmetric space $E \neq L_1$ is the intersection of two nonseparable symmetric (or, rearrangement-invariant) spaces. It is not specified by Lozanovskii whether he meant symmetric function spaces on $(0, 1)$ or on $(0, \infty)$.

This problem has been very recently resolved in the affirmative by E.M. Semenov and the reviewer. Theorem 9 in [24] yields the following result.

**Theorem 1.** *Let $E$ be a symmetric function space either on $(0, 1)$ or on $(0, \infty)$ such that $E, E^\times \neq L_1, L_1 \cap L_\infty$. There exist symmetric function spaces $E_1 \neq E$ and $E_2 \neq E$ such that $E_1 \cap E_2 = E$.*

Here, $E^\times$ is the Köthe dual of the symmetric space $E$ defined by the formula

$$E^\times = \{y \in L_1 + L_\infty : xy \in L_1, \text{ for all } x \in E\},$$
$$\|y\|_{E^\times} = \sup\{\int |xy| : \ \|x\|_E \le 1\}.$$

In [24], the nonseparability of the spaces $E_1$ and $E_2$ is not explicitly stated. However, by the construction in [24], we have $E_1 = E + M_{\psi_1}$ and $E_2 = E + M_{\psi_2}$, where $M_{\psi_1}, M_{\psi_2}$ are certain Lorentz (or, Marcinkiewicz) spaces. Since Lorentz spaces $M_{\psi_1}, M_{\psi_2}$ are nonseparable, it follows that so are $E_1$ and $E_2$.

Finally, Lemma 8 in [24] states that $E, E^\times \neq L_1, L_1 \cap L_\infty$ unless $E$ is one of the following spaces: $L_1$, $L_\infty$, $(L_\infty)_0$, $L_1 + L_\infty$, $(L_1 + L_\infty)_0$, $L_1 \cap L_\infty$.

Problem 1843 refers firstly to Makarov's proof of the following lemma. I thank Dmitriy Zanin who has suggested the following simple proof.

**Lemma 1.** *Let $f_n$, $n \geq 0$, be positive, independent and identically distributed functions. We have*

$$\sup_{n\geq 0} f_n = \|f_1\|_\infty$$

*almost everywhere.*

*Proof.* Fix a finite number $0 < M < \|f_1\|_\infty$ and consider the function $h = M\chi_{(M,\infty)}$. Clearly, the functions $h(f_n)$, $n \geq 0$, are also independent. Since $f_n \geq h(f_n)$ for every $n \geq 0$, it follows that

$$\sup_{n\geq 0} f_n \geq \sup_{n\geq 0} h(f_n) \geq \sup_{0\leq n<N} h(f_n) = M\chi_{\mathrm{supp}(\sup_{0\leq n<N} h(f_n))}.$$

By independence,

$$m\Big(\mathrm{supp}\Big(\sup_{0\leq n<N} h(f_n)\Big)\Big) = 1 - (1 - m(\mathrm{supp}(h(f_1))))^N.$$

Passing to the limit as $N \to \infty$, we obtain that

$$\sup_{n\geq 0} f_n \geq M.$$

Since $M < \|f_1\|_\infty$ is arbitrary, the assertion follows. □

If $f_n$ are positive, independent, identically distributed unbounded functions, then the positive answer to the Lozanovsky question follows by applying the lemma above to the sequence $f_{n_k}$, $k \geq 0$.

Problem 1900 (attributed to E.M. Semenov) asks whether two linearly homeomorphic symmetric spaces coincide. This innocent looking question has underlined the most deep developments in Banach space theory of symmetric function spaces. I refer to two outstanding books [11], [16], which contain a wealth of information relevant to this question. Theorem 37 in [2] answers this question in the negative via methods drawn from probability theory. The latest information concerning uniqueness of symmetric structure can be found in [3].

Problem 2099 concerns Braverman and Mekler's paper [4]. The conjecture made in this problem is resolved in the affirmative in [4]. However, this condition fails to be necessary as shown in [12]. Indeed, for every Orlicz space $L_\Phi$ the assertion stated in Problem 2099 holds true, but it is not necessarily the case that

$$\lim_{r\to\infty} \frac{1}{r}\|\sigma_r\|_{L_\Phi\to L_\Phi} = 0.$$

A necessary and sufficient condition has been recently found in [12].

Problem 551 asks whether $B_1 \wedge B_2$ is a Banach limit when $B_1$ and $B_2$ are Banach limits. Observing that the shift operator $T$ is a bijection from $l_\infty/c_0 \to l_\infty/c_0$, we infer that its adjoint $T^*\colon (l_\infty/c_0)^* \to (l_\infty/c_0)^*$ is also a bijection. It follows that $T^*$ preserves the operation $\wedge$, that is

$$T^*(B_1 \wedge B_2) = T^*B_1 \wedge T^*B_2 = B_1 \wedge B_2.$$

Hence, $B_1 \wedge B_2$ is translation invariant and positive. Hence, $B_1 \wedge B_2$ is a multiple of a Banach limit. It does not have to be a Banach limit because $B_1 \wedge B_2 = 0$

when $B_1$ and $B_2$ are distinct extreme points of the set $\mathfrak{B}$ of all Banach limits (see Lemma 1 in [25]).

One of the most useful applications of Banach limits is in the construction of singular traces [19], [21], which form an analogue of integration in A. Connes' noncommutative geometry [8]. In this context the set of almost convergent sequences is used to describe so-called measurable operators [21]. The reduction of a trace to the respective Calkin space is a symmetric functional (see [17]).

Problem 2054 asks whether the dual $M(\psi)^*$ of the Lorentz space $M(\psi)$ contains a set which consists of pairwise disjoint elements and whose cardinality is that of $\mathbb{R}$. If one replaces $M(\psi)$ with the weak $L_1$-space, $L_{1,\infty}$, then a very strong form of a positive answer exists. It is proved in [21] (see also [19]) that there is an order-preserving linear bijection between the set of all symmetric functionals on $L_{1,\infty}$ and that of all continuous translation invariant functionals on $l_\infty$ (read, multiples of Banach limits). It is proved in Lemma 1 in [25] that every 2 distinct extreme points of the set $\mathfrak{B}$ of all Banach limits are disjoint. Since (see e.g. proof of Theorem 11.1 in [20]) the set $\mathfrak{B}$ of all Banach limits has the cardinality $2^c$, it follows that there is a set of symmetric mutually disjoint functionals on $L_{1,\infty}$ whose cardinality is $2^c$.

## References


[1] Astashkin, S.V., Kalton, N. and Sukochev, F.A. (2008). Cesaro mean convergence of martingale differences in rearrangement invariant spaces. *Positivity* **12,** 387–406.

[2] Astashkin, S. and Sukochev, F. (2010). Independent functions and the geometry of Banach spaces. (In Russian.) *Uspekhi Mat. Nauk* **65,** no. 6 (396), 3–86. Translation: *Russian Math. Surveys* **65,** no. 6, 1003–1081.

[3] Astashkin, S.V., Sukochev, F.A. and Wong, C.P. (2013). Distributionally concave symmetric spaces and uniqueness of symmetric structure. *Adv. Math.* **232,** 399–431.

[4] Braverman, M. and Mekler, A. (1977). The Hardy–Littlewood property for symmetric spaces. (In Russian.) *Sibirsk. Mat. Zh.* **18,** no. 3, 522–540, 717.

[5] Carey, A.L., Gayral, V., Rennie, A. and Sukochev, F.A. (2012). Integration on locally compact noncommutative spaces. *J. Funct. Anal.* **263,** no. 2, 383–414.

[6] Chilin, V.I., Dodds, P.G., Sedaev, A.A. and Sukochev, F.A. (1996). Characterizations of Kadec-Klee properties in symmetric spaces of measurable functions. *Trans. Amer. Math. Soc.* **348,** no. 12, 4895–4918.

[7] Chilin, V.I. and Sukochev, F.A. (1994). Weak convergence in non-commutative symmetric spaces. *J. Operat. Theory* **31,** no. 1, 35–65.

[8] Connes, A. (1994). *Noncommutative Geometry.* Academic Press, San Diego, CA.

[9] Dodds, P.G., de Pagter, B. and Sukochev, F. Sets of uniformly absolutely continuous norm in symmetric spaces of measurable operators. *Trans. Amer. Math. Soc.* (to appear).

[10] Gowers, W.T. and Maurey, B. (1993). The unconditional basic sequence problem. *J. Amer. Math. Soc.* **6,** no. 4, 851–874.

[11] Johnson, W.B., Maurey, B., Schechtman, G. and Tzafriri, L. (1979). Symmetric structures in Banach spaces. *Mem. Amer. Math. Soc.* **19,** no. 217, v+298 pp.

[12] Kalton, N.J., Sukochev, F.A. and Zanin D. (2010). Orbits in symmetric spaces, II. *Studia Math.* **197,** no. 3, 257–274.

[13] Kolwicz, P., Leśnik, K. and Maligranda, L. (2014). Pointwise products of some Banach function spaces and factorization. *J. Funct. Anal.* **266,** no. 2, 616–659.

[14] Krygin, A.V., Sheremet'ev, E.M. and Sukochev, F. (1993). Conjugation of weak and measure convergence in noncommutative symmetric spaces. *Dokl. AN UzSSR* no. 2, 8-9 (in Russian).

[15] Krygin, A.V., Sheremet'ev, E.M. and Sukochev, F. (1993). Convergence in measure, weak convergence and structure of subspaces in symmetric spaces of measurable operators. Unpublished manuscript.

[16] Lindenstrauss, J. and Tzafriri, L. (1979). *Classical Banach Spaces. II. Function Spaces.* In *Ergebnisse der Mathematik und ihrer Grenzgebiete* (Results in Mathematics and Related Areas), **97**. Springer, Berlin-New York. x+243 pp.

[17] Lord, S., Sukochev, F. and Zanin D. (2013). *Singular Traces. Theory and Applications* (De Gruyter Studies in Mathematics **46**), De Gruyter, Berlin.

[18] Maligranda, L. (1989). *Orlicz Spaces and Interpolation* (Seminars in Math. **5**), University of Campinas, Campinas.

[19] Pietsch, A. (1990). Traces and shift invariant functionals. *Math. Nachr.* **145,** 7–43.

[20] Pietsch, A. (2014). Traces on operator ideals and related linear forms on sequence ideals (part II). *Integral Equations Operator Theory* **79,** no. 2, 255–299.

[21] Semenov, E., Sukochev, F, Usachev, A. and Zanin, D. Banach limits and traces on $\mathcal{L}_{1,\infty}$. Preprint.

[22] Sukochev, F.A. and Chilin, V.I. (1988). The triangle inequality for operators that are measurable with respect to Hardy-Littlewood order. (In Russian.) *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk* no. 4, 44–50.

[23] Sedaev, A.A., Sukochev, F.A. and Chilin, V.I. (1993). Weak compactness in Lorentz spaces. (In Russian.) *Uzb. Math. J.* no. 1, 84–93.

[24] Semenov, E.M. and Sukochev, F.A. (2014). Sums and intersections of symmetric operator spaces. *J. Math. Anal. Appl.* **414,** no. 2, 742–755.

[25] Semenov, E. and Sukochev, F. (2013). Extreme points of the set of Banach limits. *Positivity* **17,** no. 1, 163–170.

F.A. Sukochev
School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052.
Email: f.sukochev@unsw.edu.au

◇ ◇ ◇ ◇ ◇ ◇