Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 43. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, Faculty of Science and Technology, Federation University Australia, PO Box 663, Ballarat, Victoria 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 43 is 15 September 2015. The solutions to Puzzle Corner 43 will appear in Puzzle Corner 45 in the November 2015 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Floating fedora

Sammy dives from a bridge into a river and swims upstream for one hour at constant speed. She then turns around and swims downstream at the same speed. As Sammy passes under the original bridge, a bystander tells her that her hat fell into the river the moment she dived into the water. In order to retrieve her hat, Sammy continues to swim downstream at the same speed. She finally catches up to her hat when she is exactly one kilometre away from the bridge. Assuming it is constant, what is the speed of current?

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Social network

There is a group of 300 Twitter users, such that each one is following exactly one other person in the group. Prove that it is possible to find a smaller group of 100 in which no one is following anyone else.

Digit divisibility

A number is said to be *elegant* if its digit sum is divisible by eleven. How many elegant numbers are there in the set $\{0, 1, 2, \ldots, 10^{11} - 1\}$?

Square solitaire

Four pegs are initially placed on the ground so that they form a square. At each move, you may take an existing peg from some point P and move it to a new point P', as long as there is another peg at the midpoint of PP'. Is it possible to form a larger square using the four pegs after a finite number of moves?

Rational coordinates

Does there exist a sphere (i.e. the surface of a ball) in $\mathbb{R}^3$, such that exactly one point on it has only rational coordinates?

Solutions to Puzzle Corner 41

Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 41 is awarded to Jensen Lai. Congratulations!

Improbable product

Is it possible for the product of four consecutive positive integers to be equal to the product of two consecutive positive integers?

Solution by Martin Bunder: Denote the four consecutive positive integers by $n-1$, n, $n+1$ and $n+2$, and the two consecutive positive integers by m and $m+1$. If their products are equal, then we have the equation:

$$(n-1)n(n+1)(n+2) = (n^2+n)(n^2+n-2) = m(m+1).$$

Since $n-1$ and m are positive integers, it is clear that every term in the equation above is positive. There are two possible cases:

- If $n^2+n \leq m+1$, then $n^2+n-2 < m$ and the left side is strictly smaller.
- If $n^2+n > m+1$, then $n^2+n-2 \geq m$ and the left side is strictly larger.

In both cases, we have reached contradictions. Therefore the answer is no, it is not possible for the product of four consecutive positive integers to be equal to the product of two consecutive positive integers.

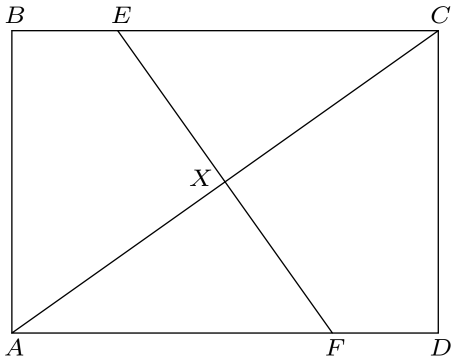
Many folds

Submitted by Andrew Kepert

(i) *An A4 paper has the length to width ratio of* $\sqrt{2} : 1$. *How many folds are needed to locate a point on the longer edge that divides the edge into the ratio of* $1 : 3$*?*

(ii) *Start with a rectangular piece of paper, choose an edge and mark a point somewhere along it. Now there are two 'far' corners which do not belong to the chosen edge. Make a fold so that one of these far corners coincides with the marked point, then unfold. Make another fold so that the other far corner coincides with the marked point, then unfold again. Prove that the intersection point of the two creases has equal distance to two opposite edges of the paper.*

Solution by Alan Jones: (i) The $1 : 3$ ratio is achievable with a single fold. Refer to the following diagram:



Let the rectangular A4 paper be $ABCD$ and denote its centre by X. Without loss of generality, let $AB = 1$ and $BC = \sqrt{2}$. Make a single fold so that the corner A coincides with the diametrically opposite corner C. Let the resulting fold line be EF as shown. We claim that E has the required property, or $BE : EC = 1 : 3$.

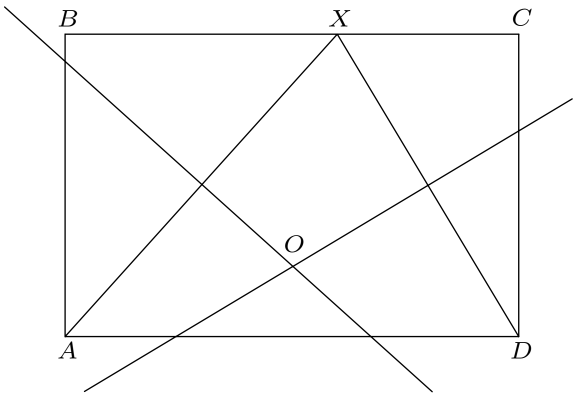
By our folding construction, EF is the perpendicular bisector of AC. So triangles ABC and EXC are similar. Hence

$$\frac{EC}{XC} = \frac{AC}{BC} \quad\Longrightarrow\quad \frac{EC}{BC} = \frac{AC^2}{2BC^2} = \frac{1^2 + \sqrt{2}^2}{2\sqrt{2}^2} = \frac{3}{4},$$

completing the solution.

Note that it is also possible to obtain the point E by making a single fold along the diagonal AC.

(ii) Refer to the diagram below:



Again let the rectangle be $ABCD$. Denote the arbitrary point along BC by X and the intersection of the creases by O. By definition, the two creases are the perpendicular bisectors of AX and DX. Since every point on a perpendicular bisector is equidistant from the two end points of the interval, we must have $AO = OX = OD$. Therefore O also lies on the perpendicular bisector of AD, so it must be equidistant from the sides AB and CD, as required. The point O is in fact the circumcentre of triangle AXD.

Rabbit season

(i) *Rachel and Fran are playing a game. Rachel controls three 'rabbit' pieces, while Fran controls a single 'fox' piece. Initially, all four pieces are placed somewhere along a straight line. They take turns making moves, with Rachel going first. Each move, a player is allowed to move one of her pieces a distance of at most one unit along the straight line. Fran wins if her fox piece can catch one of the rabbit pieces. Can Fran always win?*

(ii) *The same game is now played on a two-dimensional plane instead of a straight line. The rules are the same, except now Rachel has 20 'rabbit' pieces. Can Fran always win?*

Solution: (i) Yes, Fran can always win. Since there are three rabbits in total, two of them must lie on the same side of the fox. Fran's strategy is to move the fox 1 unit towards the two rabbits every turn until a rabbit is caught.

To see why this works, denote the fox by F and the two rabbits (which are on the same side of F) by R_1 and R_2. Consider the sum $S = FR_1 + FR_2$. Every move Fran makes will decrease S by 2, unless a rabbit is caught. But every move Rachel makes can only increase S by at most 1. Eventually, S must be no greater than 2 after Rachel's move, which means at least one of the rabbit is within 1 unit of the fox. Then Fran can win on the next move by catching that rabbit.

(ii) No, Fran cannot always win the two-dimensional version. Place the 20 rabbits on the 20 horizontal lines described by

$$y = 0, \quad y = 3, \quad y = 6, \quad \ldots, \quad y = 57.$$

Place the fox so that it is initially more than 1 unit away from all of the rabbits. For each line $y = 3i$, define its *trigger zone* to be the region that is within 1 unit of the line, or $\{3i - 1 \leq y \leq 3i + 1\}$. It is not possible to reach the line without being in its trigger zone in the previous turn. Also it is clear that the 20 trigger zones are pairwise disjoint.

Rachel's strategy is as follows: whenever the fox enters a trigger zone, move the corresponding rabbit along its horizontal line by 1 unit away from the fox. Since the fox cannot catch the rabbit the moment it enters the trigger zone, and it certainly cannot outrun the rabbit in a one-on-one chase, the fox will never be able to catch any of the rabbits.

Lighthouse logic

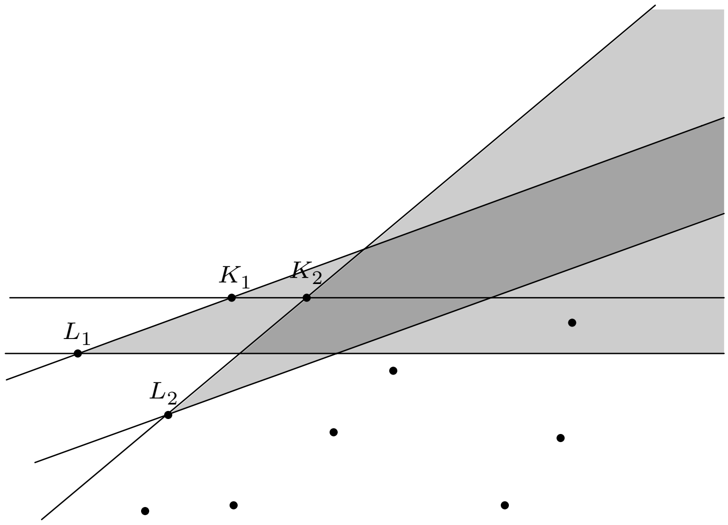
There are 18 fixed lighthouses in the plane, each has the ability to illuminate an angle of $20°$. *Prove that, by carefully selecting the directions in which the lighthouses are operating, it is always possible to illuminate the whole plane.*

Solution by Jensen Lai: Yes, it is always possible to illuminate the whole plane. For convenience, we shall assume that the 18 lighthouses are at distinct points in the plane. (The following proof still works even if two or more lighthouses are on the same point.) Also, all angles are measured anti-clockwise with respect to the positive direction of the x-axis.

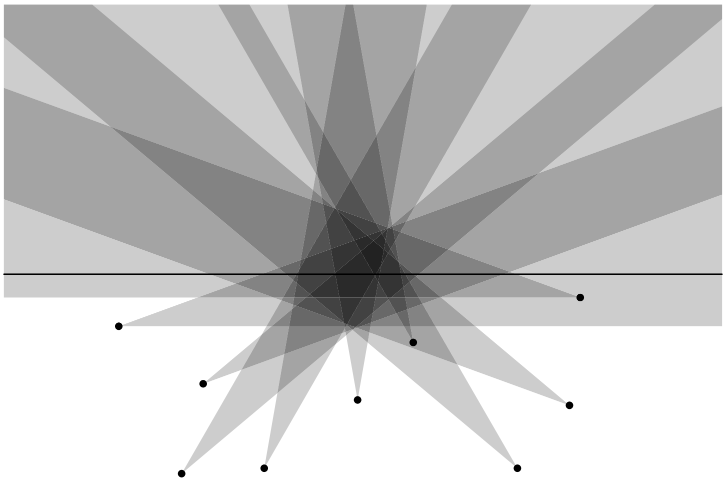
First of all, let us divide the plane into two half planes, so that each half plane contains exactly 9 lighthouses. This is possible by starting with a line and shifting it sideways until there are exactly 9 lighthouses on each side. Without loss of generality, let there be exactly 9 lighthouses in the half plane $\{y \leq 0\}$. It suffices to prove that these 9 lighthouses can illuminate the other half plane $\{y \geq 0\}$.

Draw a line through each lighthouse at an angle of $20°$. These 9 lines create 9 intersection points with the x-axis. Let the lighthouse with the leftmost intersection point be L_1 and its intersection point be K_1. Use L_1 to illuminate the angles in the range of $[0°, 20°]$. In particular, the cone with centre K_1 covering $[0°, 20°]$ is now illuminated by L_1.

Now there are 8 lighthouses remaining, all positioned to the right of the line L_1K_1. Draw a line through each of these lighthouses at an angle of $40°$, and let the 8 lines create 8 intersection points with the x-axis. Let the lighthouse with the leftmost intersection point be L_2 and its intersection point be K_2. Use L_2 to illuminate $[20°, 40°]$. Regardless of whether K_2 is on the left side or the right side of K_1, the cone with centre K_2 covering $[0°, 40°]$ is now illuminated by the combined efforts of L_1 and L_2.



Continue the same process inductively. In the ith iteration, we draw lines at an angle of $20i°$ from the unused lighthouses and let them intersect the x-axis. Let L_i be the lighthouse with the leftmost intersection point K_i and use it to illuminate $[20(i-1)°, 20i°]$. As a result, the cone with centre K_i covering $[0°, 20i°]$ is now completely illuminated by the combination of $L_1, L_2, \ldots, L_i$.



When all 9 lighthouses are lit, we have successfully illuminated the entire half plane $\{y \geq 0\}$. Repeating the same argument for the opposite half plane completes the solution.

Ivan is a Postdoctoral Research Fellow in the School of Mathematics and Applied Statistics at The University of Wollongong. His current research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.