Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 39. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, Faculty of Science & Technology, Federation University Australia, PO Box 663, Ballarat, Vic. 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 39 is 1 November 2014. The solutions to Puzzle Corner 39 will appear in Puzzle Corner 41 in the March 2015 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Integral means

Given n positive integers $a_1, a_2, \dots, a_n$, their arithmetic, geometric and harmonic means are defined as follows:

$$\begin{aligned}\text{arithmetic mean} &= \frac{a_1 + a_2 + \cdots + a_n}{n},\\ \text{geometric mean} &= \sqrt[n]{a_1 a_2 \cdots a_n},\\ \text{harmonic mean} &= \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.\end{aligned}$$

Can you find n *distinct* positive integers such that their arithmetic, geometric and harmonic means are also positive integers?

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Products of sums

We are given an $n \times n$ table where n is odd. An odd integer is written in each of its squares. Is it possible for the product of the column sums to be the negative of the product of the row sums?

Negative base

Given a positive integer $b > 1$, a *base $-b$ representation* of a number n refers to the following form:

$$n = a_0(-b)^0 + a_1(-b)^1 + \cdots + a_k(-b)^k$$

where $a_0, a_1, \ldots, a_k$ are non-negative integers less than b.

Prove that, for any positive integer $b > 1$, every integer (not just positive) has a unique base $-b$ representation.

Ranking matches

(i) Four table tennis enthusiasts are gathered to pit their skills against one another. There is a clear order in their table-tennis abilities and the better player always wins in a match. How many matches are needed to rank everyone according to their skill levels?

(ii) What if there were five table tennis enthusiasts to begin with?

Circular cuts

Submitted by Ross Atkins

The magician announces his next trick. "I have here, a piece of cardboard in the shape of a perfect circle. For my next act, I shall cut it into a number of pieces, so that all the pieces are absolutely identical to each other in shape and size. . . "

"So what", a restless audience member interjects, "anyone can do that, you've never seen a sliced pizza before?"

The magician keeps his composure. "Please let me finish. When I'm done cutting, at least one of the final pieces would not have touched the centre of the original circle to begin with."

Some started to scratch their heads. "Surely that's impossible!"

Will the magician be able to back up his words?

Solutions to Puzzle Corner 37

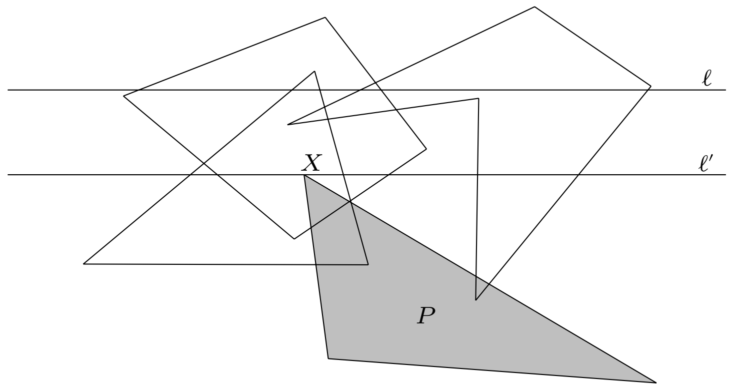
Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 37 is awarded to Joe Kupka. Congratulations!

Interesting intersection

Peter has drawn several (not necessarily convex) polygons on a piece of paper. He notices that any pair of the polygons have a non-empty intersection. Prove that Peter can draw a straight line which intersects all of the existing polygons.

Solution by Dave Johnson: For the sake of contradiction, assume that the required line does not always exist. Consider such a configuration which has the minimal number of polygons. If we delete a polygon, say P, then by minimality, there exists a line ℓ which intersects all other polygons. Furthermore, ℓ cannot intersect P.

Denote the point on P closest to ℓ by X. Construct the line ℓ' which is parallel to ℓ and passes through X.



By definition, ℓ' intersects P. Since every other polygon must intersect both ℓ as well as P, which lie on opposite sides of ℓ', it follows that every polygon must intersect ℓ', a contradiction.

Note: In fact, the same argument shows that Peter could draw a required line parallel to any direction. Also, it is not much harder to show that the same statement still holds even if there are infinitely many polygons.

Simplifying series

Simplify the following expression:

$$\frac{1}{1^4+1^2+1}+\frac{2}{2^4+2^2+1}+\cdots+\frac{100}{100^4+100^2+1}.$$

Solution by Anthony Sofo: We begin by performing the following algebraic manipulation:

$$\begin{aligned}\frac{i}{i^4+i^2+1} &= \frac{i}{(i^2+1)^2-i^2}\\ &= \frac{i}{(i^2+i+1)(i^2-i+1)}\\ &= \frac{1}{2}\left(\frac{1}{i^2-i+1}-\frac{1}{i^2+i+1}\right)\\ &= \frac{1}{2}\left(\frac{1}{(i-1)^2+(i-1)+1}-\frac{1}{i^2+i+1}\right).\end{aligned}$$

Thus the required summation can be evaluated via telescoping:

$$\begin{aligned}\sum_{i=1}^{100}\frac{i}{i^4+i^2+1} &= \frac{1}{2}\sum_{i=1}^{100}\left(\frac{1}{(i-1)^2+(i-1)+1}-\frac{1}{i^2+i+1}\right)\\ &= \frac{1}{2}\left(\frac{1}{0^2+0+1}-\frac{1}{100^2+100+1}\right)\\ &= \frac{100^2+100}{2(100^2+100+1)}\\ &= \frac{5050}{10101}.\end{aligned}$$

Cake cutting

Christie is holding a dinner party. It is known that either X or Y guests will attend. In preparation, Christie would like to cut a cake into some number of pieces (not necessarily of equal size), so that the cake can be equally shared between the guests in either scenario.

(i) *If X and Y are relatively prime, what is the minimal number of pieces required to achieve this?*

(ii) *What if X and Y are not relatively prime?*

Solution: To improve the notation slightly, let us use X and Y to denote the two potential groups of guests, and use x and y to denote the corresponding sizes of the groups. We claim the answer is $x+y-\gcd(x,y)$. For convenience, let the cake have size xy. So that if group X shows up, each guest will enjoy a slice of size y. Whereas if group Y shows up, then each guest will have a slice of size x.

Consider any scenario in which Christie has finished cutting the cake in a manner suitable for both groups. Furthermore, being an organised host, she has put name tags on the cake slices to indicate who they potentially belong to. Then each slice of cake would have exactly two names on it, one guest from X and another from Y. Without loss of generality, we may assume that any pair of names appears on at most one slice of cake (otherwise the repeating slices can simply be merged to achieve this).

Now construct a graph of $x+y$ vertices, where the vertices represent the guests from the two groups. Each edge of the graph represents a cake slice, joining the two vertices indicated by the names appearing on the slice. Since everyone receives some cake, every vertex must belong to at least one edge.

Consider any particular connected component in this graph, say it contains a vertices from X and b vertices from Y. The edges in this connected component precisely represent the cake shared by either the a guests from X or the b guests from Y. Computing the total size of these slices, we have

$$ay = bx \qquad \Longleftrightarrow \qquad \frac{a}{b} = \frac{x}{y} = \frac{x/\gcd(x,y)}{y/\gcd(x,y)}.$$

In particular, a must be a multiple of $x/\gcd(x,y)$. Since this holds for any connected components of the graph, and the various values of a must sum up to x, we conclude that there are at most $\gcd(x,y)$ connected components.

If $\gcd(x,y)=1$, then the entire graph must be connected. It is easy to see that, in order to connect a graph of $x+y$ vertices, we require at least $x+y-1$ edges. A similar argument shows that if there are at most $\gcd(x,y)$ connected components, then there must be at least $x+y-\gcd(x,y)$ edges. Recalling that the edges biject to the slices, it implies Christie needs at least $x+y-\gcd(x,y)$ slices of cake.

It remains to provide a construction for exactly $x+y-\gcd(x,y)$ slices. For the sake of argument, let the cake be a $1 \times mn$ rectangle. Christie simply measures along the length of the cake, mark out all multiples of x and y, and make cuts across the cake at those points. The multiplies of x and y only coincide at multiples of $\operatorname{lcm}(x,y)$. We arrive at the correct number of slices due to the well-known identity

$$\gcd(x,y) = \frac{xy}{\operatorname{lcm}(x,y)}.$$

Therefore $x+y-\gcd(x,y)$ slices are needed to satisfy the appetites of both groups.

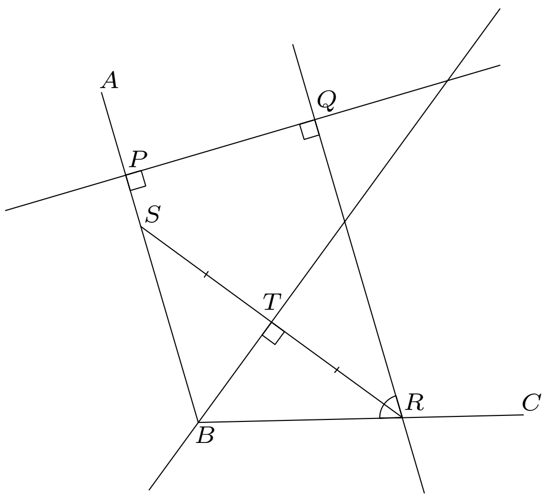
Baffling bisection

Given an angle $\angle ABC$, it is well-known that we can construct its angle bisector using only a compass and a straight edge. For the reader's interest, the steps are as follows.

- *Draw a circle centred at B, let it intersect the segments AB and AC at points P and Q respectively.*
- *Draw two (fairly large) circles of equal radii with centres at P and Q. Let these two circles intersect at points X and Y.*
- *The line XY is the required angle bisector.*

In particular, note that the point B was used in the first step. Is it still possible to construct the angle bisector of $\angle ABC$ if the point B is not allowed to be used at all?

Solution by Joe Kupka: We shall construct the bisector without ever letting B touch either ends of the compass. Refer to the following diagram.



Here are the steps in detail:

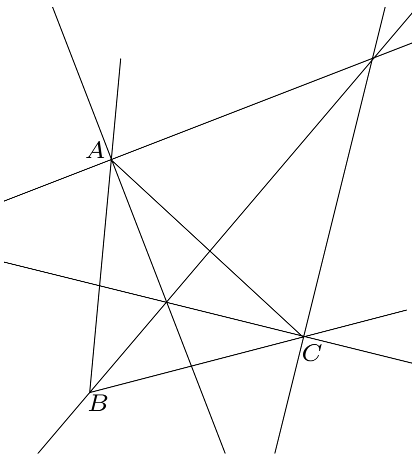
- Begin by choosing any point P on AB and construct a line through P perpendicular to AB. Constructing a perpendicular uses the same technique as bisecting an angle, except the angle here is $180°$.
- Now choose a point Q on the recently drawn line and construct new line through Q perpendicular to PQ. Let it intersect BC at R.
- Bisect the $\angle BRQ$, and let the bisector intersect AB at S.
- Construct the perpendicular bisector of SR. This can be done by drawing two (fairly large) circles of equal radii with centres at S and R, then joining the two points of intersection by the two circles. The perpendicular bisector is also the required angle bisector of $\angle ABC$.

To briefly explain the construction, note that the line QR is, by definition, parallel to AB. Then

$$\angle BSR = \angle SRQ = \angle SRB,$$

so triangle BSR is isosceles. Thus by symmetry, the perpendicular bisector of SR must also bisect $\angle ABC$.

Note: Of course there exist many alternate solutions. Here is a diagram depicting a solution which involves constructing four angle bisectors about the points A and C. See if you can iron out the details!



Ivan is a Postdoctoral Research Fellow in the School of Mathematics and Applied Statistics at The University of Wollongong. His current research involves financial modelling and stochastic games. Ivan spends much of his spare time pondering over puzzles of all flavours, as well as Olympiad Mathematics.