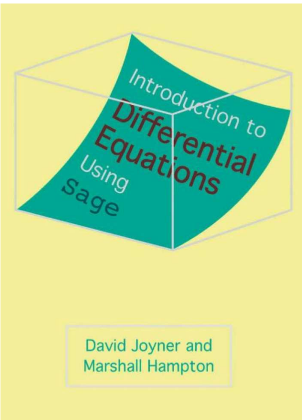
Book Reviews

Introduction to Differential Equations Using Sage

David Joyner and Marshall Hampton
The Johns Hopkins University Press, 2012, ISBN 978-1-4214-0637-4
Also available in eBook (ISBN 978-1-4214-0724-1)

Preliminary remarks

At an elementary level, there's not much new to be said about differential equations. In a first introductory course we would expect a handful of equations solvable by exact means: first-order linear, separable, Bernoulli, homogeneous, second-order linear with constant coefficients, and a sprinkling of others. And for the applications the usual suspects are trotted out: Newton's law of cooling, population growth, radiocarbon dating, mixing problems, electric circuits. Beginning students, flicking through a few texts, might reasonably wonder if this is it? Most differential equations which are used with such power in modern modelling are of course extremely complex, and systems such as the Navier–Stokes equations, Maxwell's equations, wave equations lead to some very subtle mathematics.



It is a problem for an educator in such an introductory course to find examples which are simple to define and describe, solvable, if not exactly, then with reasonable accuracy by simple numerical means. And an educator's job is made much easier if everybody has access to a computer algebra system. Not only can the system (if it's powerful enough) solve a large class of equations analytically, but also numerically. The system should also be able to draw graphs: of solutions, of direction fields, of different approximations to a solution.

For this text the authors have chosen Sage. Sage is somewhat of a new kid on the block, being less than ten years old. Initially it consisted of mostly other free and open source (FOSS) mathematical software, all pulled together with a consistent interface using the language Python. Since its inception it has grown hugely, has had an enormous amount of code written for it, and for algebraic geometry and number theory is probably the most powerful software in current use, with much of its code being contributed by some of the leading researchers in their fields. For calculus related material, Sage devolves most of its working to Maxima, which is the current FOSS version of the venerable system Macsyma, which is no longer in use. Maxima is part of Sage; when you download and install Sage, you also get a complete Maxima.

The authors are both leading contributors to the Sage project, and the first author has worked closely with the lead developer, William Stein from the University of

Washington, from the very beginning. They are also both exemplary mathematicians and educators.

The Book

The book started out as course notes by the first author, which were then revised and extended by the second. The book still seems to me to have the feeling of course notes, rather than a text book, and although I can't fault its content or much of its exposition, there are places where the lack of some good independent editing is apparent.

First, the mathematical content. The book consists of four chapters: First Order DEs, Second Order DEs, Systems of DEs, and PDEs. Such numerical examples as are given are placed in the context of the equations, rather than as a separate section or chapter. And the first chapter contains most of the numerical material, although later chapters include some numerical examples.

Each chapter has a good mix of theory, Sage examples, and exercises. Exercises are given at the end of each section, and vary between pencil and paper exercises, Sage exercises, and applications. Sage examples are copiously scattered throughout the text, but there is a tendency to introduce Sage commands without explaining them first. I think the text needs either an introductory chapter describing how to download and install Sage, or how to access it on one of its public servers. I also think that the Sage examples would look nicer if they somehow simulated the notebook interface, where output can be mathematically typeset using LaTeX.

Occasionally there are typos which would have been picked up in an editing process. For example, on page 27, the Sage example defines a variable t, a function $x(t)$, but the DE is then given in terms of $y(t)$, following the example given just previously. Then the equation is solved for $x(t)$. This is confusing.

Occasionally also the book presents some enticing mathematics without any further development. For example, both Peano's and Picard's theorems for the existence of solutions are given very early on, but the authors seem to skate over the important fact that replacing continuity (for Peano's theorem) with Lipschitz continuity (for Picard's theorem) provides for the existence of a unique solution. Then we never hear any more about them. If such important results are to be included at all, I think they need more than a cursory page. In fact, a chapter devoted to some of the theory of DEs, as opposed to methods of solution, would be a wonderful addition. Maybe in the next edition. . . ?

I am biased, but I wish there was a little more detail on numerical solutions, and some examples of the use of Sage's `ode_solver()` command, which includes a Runge–Kutta–Fehlberg (RKF) adaptive stepsize method. The authors allude to such a method on page 50, but instead of using it, a function is given which compares a fourth-order Runge–Kutta method with a second-order improved Euler method. In fact in general for an adaptive stepsize approach the two methods used differ by an order of one, thus the RKF method compares the results of a fourth order method and a fifth-order method. It should also be noted that numerical methods for systems may differ from those used for a single equation. An Adams–Bashforth method is given without any approach at derivation, although in fact it's very easy, using the interface to Maxima and Maxima's `interpol` package for interpolation:

```
Sage

sage: maxima('load(interpol)')
sage: p = maxima('lagrange([[x0,f0],[x0+h,f1],[x0+2*h,f2],\
....:   [x0+3*h,f3]],x)')
sage: p.integrate(x,x0+3*h,x0+4*h).expand().factor()
```

$$(55f_3 - 59f_2 + 37f_1 - 9f_0)\frac{h}{24}$$

This is the term to be added to y_3 to obtain y_4, and can be seen to be the second term in equations 1.11 on page 49 (for $n = 0$).

In Chapter 2, in discussing the solution of the characteristic equation for a linear DE with constant coefficients, there is the following curious inclusion:

> To solve the homogeneous differential equation in (2.1), please *memorize these rules*
>
> real $r \Rightarrow e^{rt}$
> complex $\alpha + \beta i \Rightarrow e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)$
> repeated roots $\Rightarrow$ repeated solutions, boosted each time by t

Now I'm sure we have all recommended that students memorize a few simple rules; and in a classroom this would be fine. But in a textbook? Why these particular rules and no others? There are no instructions to memorize, for example, the integrating factor technique for first-order linear equations. This is one of those places where there is an apparent lack of external editing.

This is the chapter in which a lack of consistency with the diagrams becomes apparent. Some diagrams seem copied as raster images from elsewhere and seem a bit blurred (figure 2.14 on p. 106); others (such as graphs of functions) are directly produced by Sage and copied in. But to give the best appearance in a LaTeX document the authors should have used a proper vector package, such as TiKZ, for all their figures. This would not only greatly improve the appearance of figures, but give some measure of homogeneity between them. On pages 184–186 for example, there are three circuit diagrams, all with different fonts and circuit elements; to my mind this looks sloppy. I think in a textbook every attempt should be made to make the book look as professional as possible, and this includes not only the elegance and correctness of the mathematics, but also of the diagrams.

In Chapter 3 (Matrix theory and systems of DEs) a full 25 pages is taken up with elementary linear algebra. I imagine the authors were keen to ensure that the book was as self-contained as possible, and they assumed that readers would have had some basic calculus, but no linear algebra. I think a better use of space would have been to present a very cut-down introduction, maybe including references to online material, and spend more time on the DEs, in particular on numerical solutions of systems. This chapter does finish with a splendid example of the now famous 'zombie attack' model, which is a delightful version of the standard SIR model for disease spread. A number of examples are given with different parameters showing how in some cases the 'zombies win' and in others the 'zombies lose'. (And in two of the three diagrams 'susceptibles' is mis-spelled 'suseptibles'.) More space might have allowed the authors to show how the parameters can be determined (by a least squares approach) to fit particular data. As with the previous two chapters there are some initial value problems given, but no boundary value problems.

Chapter 4 (PDEs) contains material relating to trigonometric series, and shows how these can be used to solve the wave equation and Schrödinger's equation. What is missing is the derivation of these equations, or much discussion of what they may mean as models of the physical world. For this reason this chapter seems a bit superficial. Also, as the authors point out, Sage's current ability to solve PDEs is very limited. I think that this chapter would have been better replaced with an introduction to Sage. Then there would have been also more room to investigate other DEs, such as Newton's gravitational equations and the derivation of Kepler's laws, or use Sage to take the students further afield with other DEs, such as Riccati equations.

I also have some concerns with the bibliography, which consists almost wholly of wikipedia articles. This is a contentious point: I myself love wikipedia, and I think that most of its mathematics is exemplary. But wikipedia is not peer-reviewed, and errors can creep in, sometimes with an edit, sometimes of fact. For this reason, although it is a superb resource, I don't believe it can take the place of reviewed and professionally edited published material. Possibly there should be two bibliographies, one of published material, and the other of non-reviewed online material (such as wikipedia) with a stern warning that such material may not necessarily be always correct.

Finally there is no website associated with the book at which you can download the Sage examples and functions. This is an omission which should be rectified.

Conclusions

In many ways this is a fine book: it contains pretty much all of the material one would expect to find in a first, introductory DEs course, and provides a nice mix of theory and symbolic computation. The pedagogy is excellent and the exercises are models of their kind. (I am always grumbling about the poor quality of exercises in mathematics texts, and for once I can't!) It is let down only by what seems to be to be hastiness in its publication, which has allowed some typos and inconsistencies of exposition to slip through. I hope this book sees a second edition — it deserves one — in which the text is tightened by appropriate editing.

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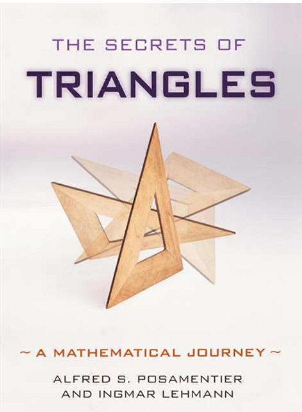
The Secrets of Triangles: A Mathematical Journey

Alfred S. Posamentier and Ingmar Lehmann
Prometheus Books, 2012, ISBN 978-1-6161-4587-3

Preliminary remarks

Euclidean geometry is one of those topics which seems to have disappeared from modern mathematics curricula. Although we all learn cartesian geometry very early on, the mathematics of Euclid and his contemporaries, and of more recent geometers (Poncelet, Steiner) is barely touched on. Partly this is due to the axiomatization of recent mathematics and of the more central role played by algebra; indeed Hilbert's approach was to denigrate the pictorial aspect of geometry; as he said in the very late 19th century: 'One must be able to say at all times — instead of points, straight lines, and planes — tables, chairs, and beer mugs.'[1] And Hilbert's view has coloured geometry ever since. I think this is a pity: Euclidean geometry, with its points and lines (sorry, Hilbert!) and diagrams has a charm all of its own, and one without which mathematics would be immeasurably poorer.



In spite of Hilbert and of the huge advances made by algebraic geometry, there has always been a steady undercurrent of Euclidean geometry, and new discoveries are constantly being made, with new theorems and proofs. Some of these results would fit nicely into an expanded Euclid; others require more advanced methods, using results from algebra, such as Gröbner bases.

In particular, there is a constant stream of new 'triangle centres'. Possibly everybody is familiar with at least three: the *centroid*, which is the point of concurrency of the triangle medians (lines between vertices and mid-points of opposite sides); the *incentre*, which is the centre of the inscribed circle, and point of concurrency of the angle bisectors; and the *circumcentre*, which is the centre of the circle through the vertices, and point of concurrency of the perpendicular bisectors of the sides. Figure 1 shows them all, with M being the centroid, and I, E in the incentre and circumcentre respectively.

What is perhaps less well known is that there are (at the time of writing) nearly *six thousand* triangle centres, all of which are described at the *Encyclopedia of Triangle Centers* at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html. This text by Posamentier and Lehmann may be considered as an enticement into this world.

[1] *Hilbert*, Constance Reid, Springer, 1996

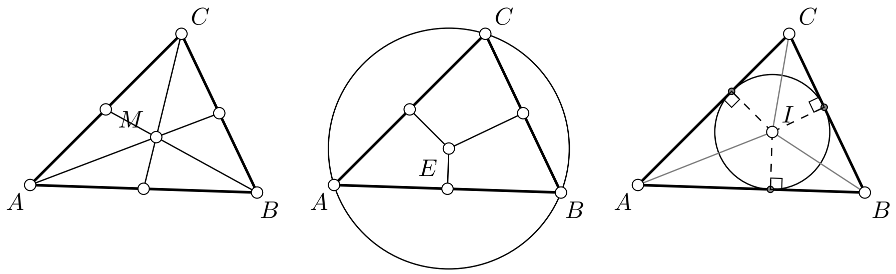


Figure 1: Three standard triangle centres

The book itself

The authors are well known as mathematics educators and writers of books for the general mathematical public, and this book (as its title suggests) concentrates on triangles, mainly from a Euclidean perspective. The copy I have for review is an 'Uncorrected Advance Reading Copy', which is not edited, and lacks an index, as well as a comprehensive bibliography. The unedited nature means that sometimes the caption of a diagram is on the next page, which makes the text harder to follow. Each diagram (in this copy) is accompanied by its file name, and thus it can be seen that the authors have used Geometers Sketchpad to create them. Although this software (and Geogebra) are mentioned, there is no discussion as to how they can be used to help understand the material. One problem with diagrams in geometry is that they can get complicated very easily, and the reader has to work hard to untangle the messy web of points, lines and circles. One such figure is 6.22, showing how the nine-point circle is also the nine-point circle of another triangle formed by the centres of circles. Many of the diagrams in this chapter are highly convoluted. It's interesting to turn to the great text of Coxeter[2] and to look at his diagrams, in which all extraneous material is removed. For the nine-point circle, Coxeter shows the points, and a minimal construction required for his proof, but not the circle itself. This makes for a very clear and readable diagram. The use of dynamic geometry software (Geometers Sketchpad, Geogebra, and others) is a real boon for students (and anybody else), as you can use different colours for different lines and curves, and of course pull points around and see what happens to the diagram.

I think it is a pity that a text like this does not make more use of such software, or provide a companion website where the material is available in files for the most common systems. A diagram which is complicated on the page can be rendered with greater clarity on a computer screen, with the addition of colours, line thicknesses, and placement of points.

Posamentier and Lehmann are keen to introduce the reader to much of the modern theory of triangles, without bothering too much about theorems and proofs, although a few are given. Ceva's theorem, which states the conditions under which

[2] *Geometry*, 2nd edn, H.S.M. Coxeter, Wiley, 1989

lines from vertices to the opposite sides are concurrent, is proved, and the concurrency of medians follows immediately. Morley's theorem, that the interior angle trisectors meet at the vertices of an equilateral triangle, has its proof given in an appendix (which is missing from my copy).

With nearly 6000 centres to choose from, the authors concentrate naturally on the most 'natural'; that is, the ones whose construction is fairly simple. The discussion of such centres makes up much of the first half of the book. The authors discuss centres obtained by concurrency of lines, and those obtained by concurrency of circles. The nine-point circle theorem is proved, in a somewhat long-winded fashion (in comparison, Coxeter's neat, exemplary proof takes about half a page).

The second half of the book looks at areas, constructions, inequalities, and finishes off with a pleasant little chapter about fractals, where as you would expect the Sierpinski Gasket (also known as the Sierpinski Triangle), and Koch's Snowflake Curve, are discussed, along with some others.

The chapter on areas gives Heron's formula — as you would expect — but not its proof, which is relegated to the (missing) appendix. I would have thought that such an elegant result would be worthy of a proof in the body of the text. This chapter also includes plenty of trigonometry, and the sine and cosine rules, for example, can be used to provide a very quick and straightforward proof.

On the subject of proofs, the authors have deliberately shied away from the formal lemma-theorem-proof style of mathematical writing to a more discursive approach. Pythagoras' theorem is in fact proved using similar triangles, but the authors discuss similar triangles and some relationships between them, out of which the theorem falls as a sort of corollary. This may be a fine style of writing, but I found it hard to make my way around in it.

The chapter on inequalities is a nice inclusion, and concentrates on inequalities relating to lines in the triangle, including the Erdős–Mordell inequality: for any point P inside a triangle ABC the sum of distances from P to the sides is less than or equal to half the sum of distances to the vertices (with equality when P is the centroid). Given the previous chapter on areas, I was sad not to see my favorite triangle inequality: Weitzenböck's inequality, which relates the area Δ of a triangle to its sides as $a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta$. However, one single book can't contain everything.

I also take exception to the authors' occasional breathless excitement; when a result is described as being 'amazing', 'wonderful', 'cute', 'fantastic', and 'nifty'. Heron's theorem is a 'nifty result'? It's certainly elegant, and if you're new to this area of mathematics, perhaps remarkable, but 'nifty'?

Conclusions

Who is this book for, and who would gain the most from it? A professional mathematician would not find much of interest here; the book is too discursive, and the various proofs are hard to track down. On the other hand, it *is* a nice compendium of many elementary triangle results, and would make for an enjoyable bedside read.

I think the book would be best suited for high school students, or the general interested public wishing to have a glance at this most enticing and accessible corner of mathematics.

It may be considered as a sort of extension to David Wells' *The Penguin Book of Curious and Interesting Geometry* (1992), which although having no proofs at all, does have some very elegant diagrams. Posamentier and Lehmann's book aims to be as chatty and interesting, yet also with a few proofs and a greater sense of formal enrichment.

I'm sure that the final published book, with properly named and placed diagrams, with the mysterious proof-rich appendix, and a decent bibliography and index, will go a long way to countering some of my niggles, and will deserve a place on the shelves of any mathematical library.

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