Lift-Off Fellowship report
Polytopal realisations of cluster complexes

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As a Lift-Off Fellow, I have had the opportunity to continue doing research after submitting my PhD. The proposed area of research of my Lift-Off Fellowship was to investigate questions relating to polytopes proposed in the paper [2]. An important question they ask, is whether a particular family of simplicial complexes, which are known to be spherical, can be realised as the boundary of a polytope. I have been focussing on this question, and in the process I have been able to greatly expand my knowledge base. Since the completion of my PhD and being awarded the fellowship, I have also been able to conduct some research into permutation patterns in the paper [1]. In this article, I summarise the main definitions and results leading to the questions asked in [2], as well as summarising my results from [1].

A *cluster complex* is a type of simplicial complex, which was originally defined by Fomin and Zelevinsky in [3]. Their definition was then extended so that it corresponded to all Coxeter groups by Reading in [6] [7]. *Subword complexes* are a family of simplicial complexes that were first defined by Knutson and Miller in [4], who then extended their definition to include all Coxeter groups in [5]. In [2], the authors Ceballos, Labbé and Stump describe a one-to-one correspondence between cluster complexes and a subset of the subword complexes, and then generalise cluster complexes to *multi-cluster complexes* by using the subword complex description. They introduce a parameter k, so that when $k = 1$, the multi-cluster complexes reduce to cluster complexes. The following definitions appear in [2].

A *finite Coxeter system* is a pair (W, S) where S is a finite set, and W is a finite group with presentation

$$\langle s \in S \mid (ss')^{m_{ss'}=1} \rangle,$$

with $m_{ss} = 1$ and when $s \neq s'$ we have $m_{ss'} \in \{2, 3, 4, \dots\} \cup \{\infty\}$. A group with such a presentation is called a *Coxeter group*, and it is a well-known fact that finite Coxeter systems biject to finite Coxeter groups.

Given a finite Coxeter system (W, S), let Q be a word in the generators S of W, and let $\pi \in W$. The *subword complex* $\Delta(Q, \pi)$ is the simplicial complex whose vertices are the letters in Q (there is a vertex corresponding to each copy of a generator in Q, even if it appears more than once), and whose faces are subwords P of Q such that $Q \setminus P$ contains a reduced expression for π. In [4], Knutson and

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Miller show that subword complexes are homeomorphic to either a sphere or a ball.

If (W, S) is a finite Coxeter system, then a *Coxeter element* c is a product of the generators S in some order. We let w_0 denote the longest element of the Coxeter group (which is known to be unique), and we let $\boldsymbol{w_0}(\boldsymbol{c})$ denote the *c-sorting word* for w_o, which is the lexicographically first subword of $\boldsymbol{c}^\infty$ that represents a reduced expression for w_0. Then the *multi-cluster complex*, denoted Δ_c^k, is the subword complex:

$$\Delta_c^k := \Delta(\boldsymbol{c^k w_0}(\boldsymbol{c}), w_0).$$

The following interesting facts are known about multi-cluster complexes:

- For any Coxeter group W, any integer k, and any two Coxeter elements c, c', the complexes $\Delta_c^k(W)$ and $\Delta_{c'}^k(W)$ are isomorphic.
- When $W = A_{m-2k-1}$, the complex $\Delta_c^k(A_{m-2k-1})$ is isomorphic to the simplicial complex whose facets correspond to k-triangulation of a convex m-gon, which is denoted $\Delta_{m,k}$ (see [2]). When $k = 1$, this is the dual simplicial complex to the associahedra.
- When $W = B_{m-k}$, the complex $\Delta_c^k(B_{m-k})$ is isomorphic to the simplicial complex whose facets correspond to centrally-symmetric k-triangulation of a regular convex m-gon, which is denoted $\Delta_{m,k}^{\text{sym}}$ (see [2]).

It is known that the simplicial complex $\Delta_{m,1}$ can be realised as the boundary complex of a polytope. However, it is not known whether this is true for multi-cluster complexes, or even the families $\Delta_{m,k}$ and $\Delta_{m,k}^{\text{sym}}$ in general. Ceballos, Labbé and Stump conjecture that the multi-cluster complex is the boundary complex of a simplicial polytope, [2, Conjecture 9.9]. The same question was asked for the cluster complexes in [4, Question 6.4]. Solving this question has been the main focus of my research for the Lift-Off Fellowship.

On top of my research into finding a polytopal realisation of cluster complexes, I have been investigating pattern popularity within 132-avoiding permutations. A permutation σ *contains* the permutation τ if there is a subsequence of σ order isomorphic to τ. A permutation σ is *τ-avoiding* if it does not contain the permutation τ. For any n, the *popularity* of a permutation τ is the number of copies of τ in the set of all 132-avoiding permutations of length n. Rudolph shows in [8] that 132-avoiding permutations of length k with the same spine structure are equally popular within 132-avoiding permutations of any length n. Rudolph conjectures in [8, Conjecture 21] that for permutations τ and μ of the same length, the popularity of τ is less than or equal to the popularity of μ, if and only if the spine structure of τ is less than or equal to the spine structure of μ in refinement order. I was able to prove one direction of this conjecture in [1], by showing that if the spine structure of τ is less than or equal to the spine structure of μ, then for all n, τ is less popular than μ. I was able to disprove the opposite direction by giving a counterexample, and hence disprove the conjecture.

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Natalie has recently completed a PhD at the University of Sydney under the supervision of Anthony Henderson. Her thesis was titled *The Gamma-Polynomial of Flag Homology Spheres* and was conducted in the area of geometric combinatorics. She was awarded the 2013 T.G. Room Medal for a PhD thesis in pure mathematics of outstanding merit. Previous to this, she completed a Masters by research at the University of Melbourne under the supervision of Arun Ram. Her master's thesis was titled *Duality Groups*, and focussed on a new classification of duality groups, which are a family of complex reflection groups. Natalie completed her undergraduate degree with honours in Applied Science at RMIT University.