



Book Reviews

Convex functions: Constructions, characterizations and counterexamples

Jonathan M. Borwein and Jon D. Vanderwerff
Springer, 2011, ISBN 978-0-387-87856-0

I think the best way to begin is to quote some introductory remarks from the book.

Like differentiability, convexity is a natural and powerful property of functions that plays a significant role in many areas of mathematics, both pure and applied. It ties together notions from topology, algebra, geometry and analysis, and is an important tool in optimization, mathematical programming and game theory. This book, which is the product of a collaboration of over 15 years, is unique in that it focuses on convex functions themselves, rather than on convex analysis. The authors explore the various classes and their characterizations, treating convex functions in both Euclidean and Banach spaces. . . .

The book can either be read sequentially as a graduate text, or dipped into by researchers and practitioners. Each chapter contains a variety of concrete examples and over 600 exercises are included, ranging in difficulty from early graduate to research level.

I am, I suppose, a *researcher and practitioner*. I am not an expert on convexity but I use it continually. I have always wished I knew more but have never quite had the energy or drive to make myself both learn and understand the topic in any great depth. Until now, my most successful attempt—which was a major step forward for me—was to purchase the book *Convex Analysis and Nonlinear Optimization: Theory and Examples* by Jonathan M. Borwein and Adrian S. Lewis. To date, I have read and become quite familiar with around the first third of this book—which incidentally is a delight to read and study—and that is essentially my starting point for this review. When the opportunity arose to review the new book I was both unable to resist and yet reluctant to be drawn in. Unable to resist because I have really enjoyed my reading of the first book but reluctant because the new book promised to be more comprehensive and much deeper. And here was I, in my *spare* time, still struggling through the original. I love thinking and talking about mathematics but I am a reluctant reader. So—what have I found? My first observation is that the new book rests comfortably on the previous book by Borwein and Lewis. At least in the first



three chapters there are frequent references to the earlier work and although there is inevitably common ground it is, to my mind, deftly managed.

The first chapter introduces the basic properties of convex functions. The concept of an epigraph—the region above the graph—is fundamental as is the idea of a subdifferential. If E is a Euclidean space and $f: E \rightarrow [-\infty, +\infty]$ then the epigraph is the set $\text{epi } f := \{(x, t) \in E \times \mathbb{R} \mid f(x) \leq t\}$. If $\text{epi } f$ is closed in $E \times \mathbb{R}$ then we say that f is closed. The domain of f is the set $\text{dom } f = \{x \in E \mid f(x) < \infty\}$. The *subdifferential* of f at $\bar{x} \in \text{dom } f$ is the set

$$\partial f(\bar{x}) := \{\phi \in E \mid \langle \phi, y - \bar{x} \rangle \leq f(y) - f(\bar{x}), \forall y \in E\}.$$

If $\bar{x} \notin \text{dom } f$ then the subdifferential is the empty set. That is $\partial f(\bar{x}) = \emptyset$. If $\partial f(\bar{x}) \neq \emptyset$ and $\phi \in \partial f(\bar{x})$ then ϕ is a subgradient for f at \bar{x} . The *indicator function* of a nonempty set $C \subset E$ is the function δ_C defined by

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

If C is a convex set then δ_C is a convex function. The indicator function helps us to talk interchangeably about convex sets and convex functions. Another important concept is the idea of a cone. Cones are used to define order relationships. A set $K \subset E$ is said to be a cone if $tK \subset K$ for every $t \geq 0$. In particular $\mathbb{R}_+ = [0, +\infty)$ is the nonnegative cone in \mathbb{R} . Carefully selected definitions, key properties and important theorems are introduced without proof but with copious references to later chapters where proofs and detailed discussions can be found. Some mathematical applications are also presented to provide context and motivation. Well-chosen mathematical applications include the Birkhoff theorem *that each doubly stochastic matrix is a convex combination of permutation matrices*, an example of a continuous but nowhere differentiable function and a discussion of the Gauss theorem *that roots of the derivative of a polynomial lie inside the convex hull of the zeros*. There are also applied examples. A first intriguing example concerns *hidden convexity* in the classic Brachistochrone problem. A later example challenges readers to prove that the mapping

$$p \mapsto \sqrt{p} \int_0^\infty \left| \frac{\sin x}{x} \right|^p dx$$

can be expressed as the difference of two convex mappings for $p \in (1, \infty)$. And there is more—from the famous von Neumann minimax result in game theory to the inevitable discussion of *entropy*. To my way of thinking this chapter is an excellent introduction which can be summed up best by once again returning to the text for illustration.

The function f is *Fréchet differentiable* at $\bar{x} \in \text{dom } f$ with Fréchet derivative $f'(\bar{x})$ if

$$\lim_{t \rightarrow 0} \frac{f(\bar{x} + th) - f(\bar{x})}{t} = \langle f'(\bar{x}), h \rangle$$

exists uniformly for all h in the unit sphere. If the limit exists only pointwise, f is *Gâteaux differentiable*.

I had to think about this for a while but soon realised it was an economic and elegant expression of the usual definition of Fréchet differentiability — that for each $\epsilon > 0$ there exists $\delta > 0$ so that

$$\left\| \frac{f(\bar{x} + th) - f(\bar{x})}{t} - \langle f'(\bar{x}), h \rangle \right\| \leq \epsilon \|h\|$$

for all h and all t with $0 < |t| < \delta$. It can be seen here that a little prior knowledge is needed to gain the full benefit of the exposition — but perhaps that is how it should be.

Chapter 2 looks at convex functions on Euclidean spaces. There are no real surprises here but the treatment is crisp and complete. We begin with continuity and subdifferentials and move on to differentiability, conjugate functions and Fenchel duality, second order differentiability and finally support functions and extremal structures. The *Fenchel conjugate* of a function $f: E \rightarrow [-\infty, +\infty]$ is the function $f^*: E \rightarrow [-\infty, +\infty]$ defined by

$$f^*(\phi) := \sup_{x \in E} \{\langle \phi, x \rangle - f(x)\}$$

from which one may deduce the important *Fenchel–Young* inequality

$$f^*(\phi) + f(x) \geq \langle \phi, x \rangle$$

for all $\phi \in E$ and $x \in \text{dom } f$ provided $f: E \rightarrow (-\infty, +\infty]$. The basic properties of convex functions are discussed in depth and there are numerous informed examples and problems — some straightforward and some challenging. The approach is clear and concise but also recognizes the later need for a more general view. Thus, for instance, we have — early on — the definition of a *balanced* set and a preliminary discussion about the broader significance of norms in Banach space. The *Minkowski functional* is introduced too and there is a Euclidean space version of the Hahn–Banach theorem. The Fenchel–Young inequality and the Fenchel duality theorem are perhaps the main focus while other key results such as the Fan minimax theorem and proof are presented (with some hints from the authors and some serious thinking required by the reader) in the many exercises. There are exercises on a classical maximum entropy problem and the generalized Steiner problem. There is an in-depth discussion of conjugate functions and Fenchel duality leading on to a Sandwich theorem and various other separation theorems. Differentiation of convex functions is investigated exhaustively and culminates in a famous Alexandrov theorem *that every convex function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ has a second-order Taylor expansion almost everywhere*.

Chapter 3 studies the finer structure of Euclidean spaces. The first topic embraces polyhedral convex sets and functions. The definitions come thick and fast here and we quickly arrive at the key theorem of polyhedrality *that a convex set or function is polyhedral if and only if it is finitely generated*. The algebra of polyhedral functions under linear transformation is outlined and the fundamental theorems on Fenchel duality are adapted to polyhedral functions. Another theme is the study of functions of eigenvalues. I need to outline some definitions to impart a sense of what this is all about. Define $\mathbb{R}_+^n = [0, +\infty)^n$ and $\mathbb{R}_{++}^n = (0, +\infty)^n$. Let S^n denote the set of real $n \times n$ symmetric matrices with subsets S_+^n and S_{++}^n of positive semidefinite and positive definite matrices respectively. There is

a partial order on S^n written as $X \preceq Y$ if $X, Y \in S^n$ and $Y - X \in S_+^n$. The mapping $\lambda: S^n \rightarrow \mathbb{R}^n$ is defined by the vector $\lambda(A) \in \mathbb{R}^n$ whose components are the eigenvalues of A in nonincreasing order. The space S^n becomes a Euclidean space with the definition $\langle X, Y \rangle = \text{tr}(XY)$ where $\text{tr}(A)$ denotes the trace of A . The key to this study of functions of eigenvalues is Fan's theorem that *all real symmetric matrices $X, Y \in S^n$ satisfy the inequality*

$$\text{tr}(XY) \leq \lambda(X)^T \lambda(Y) \Leftrightarrow \langle X, Y \rangle \leq \langle \lambda(X), \lambda(Y) \rangle$$

with equality if and only if there is a simultaneous ordered spectral decomposition defined by an orthogonal matrix U such that $X = U^T[\text{diag } \lambda(X)]U$ and $Y = U^T[\text{diag } \lambda(Y)]U$. A special case yields the classical inequality $x^T y \leq [x]^T [y]$ where $x, y \in \mathbb{R}^n$ and $[x]$ denotes the vector obtained from x by rearranging the components into nonincreasing order. Now we can say that $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is *symmetric* if $f(x) = f([x])$. These definitions enable the authors to state some beautiful and fascinating results. First is a result about barriers. *The functions $lb: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ and $ld: S^n \rightarrow (-\infty, +\infty]$ defined by*

$$lb(x) := \begin{cases} -\sum_{i=1}^n \log x_i & \text{if } x \in \mathbb{R}_{++}^n \\ +\infty & \text{otherwise} \end{cases}$$

and

$$ld(X) := \begin{cases} -\log \det X & \text{if } X \in S_{++}^n \\ +\infty & \text{otherwise} \end{cases}$$

are essentially smooth and strictly convex on their domains and they satisfy the conjugacy relations $lb^*(x) = lb(-x) - n$ for all $x \in \mathbb{R}_+^n$ and $ld^*(X) = ld(-X) - n$ for all $X \in S_+^n$. The vector and matrix examples can be related through the identities

$$\delta_{S_+^n} = \delta_{\mathbb{R}_+^n} \circ \lambda \quad \text{and} \quad ld = lb \circ \lambda.$$

Second is a theorem about spectral conjugacy. *If $f: \mathbb{R}^n \rightarrow [-\infty, +\infty]$ is a symmetric function it satisfies the formula $(f \circ \lambda)^* = f^* \circ \lambda$.* Next are a sequence of interesting corollaries on spectral functions, symmetry and convexity.

The authors move on to talk about linear and semidefinite programming duality and to present some results concerning selection and fixed point theorems. The chapter concludes with a section that looks *into the infinite*. The idea is to present the Euclidean space *avatars* of some important results from later chapters that are usually stated in more general settings—locally convex spaces, normed spaces, Banach spaces and Hilbert spaces. These include the Ekeland variational principle and some well-known results about convex functions as well as a brief introduction to the remarkable *Fitzpatrick function* $\mathcal{F}_T: E \times E \rightarrow \mathbb{R}$ defined for a multifunction T by the formula

$$\mathcal{F}_T(x, x^*) := \sup\{\langle x, y^* \rangle + \langle x^*, y \rangle - \langle y, y^* \rangle \mid y^* \in T(y)\}.$$

As a final note the authors provide a guide on *how to browse in the rest of the book*.

I will not describe the rest of the book in detail. The essence of the story has already been told. Suffice it to say that what remains—and there is a vast amount that does remain—is as much a reference library as it is a story, with a plethora of

extensions, intricacies and exceptions. The chapter titles are *Convex functions on Banach spaces* where my old favourite Conditional Value-at-Risk (CVaR) makes a guest appearance in Problem 4.4.24; *Duality between smoothness and strict convexity*; *Further analytic topics* which includes sections on multifunctions and monotone operators, an introduction to epigraphical convergence, convex integral functions, strongly rotund functions, trace class convex spectral functions, deeper support structures, and convex functions on normed lattices; *Barriers and Legendre functions*; *Convex functions and classifications of Banach spaces*; *Monotone operators and the Fitzpatrick function*; and lastly some *Further remarks and notes* where the authors examine the role of finite dimensionality and ponder the essential differences between convex functions on Euclidean, Hilbert and Banach spaces. More information with additional notes, some solutions and errata can be found at <http://carma.newcastle.edu.au/ConvexFunctions/>.

This is a comprehensive reference book from two experts in the field. Those parts that I have read in detail are concisely written and informative but also interesting and challenging. I cannot easily imagine a better book on this topic. It should become an indispensable reference work for both experts on convex functions and those of us who simply wish to apply and briefly understand these remarkable results from time to time.

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Wizards, Aliens and Starships

Charles L. Adler

Princeton University Press, 2014, ISBN 978-0-691-14715-4

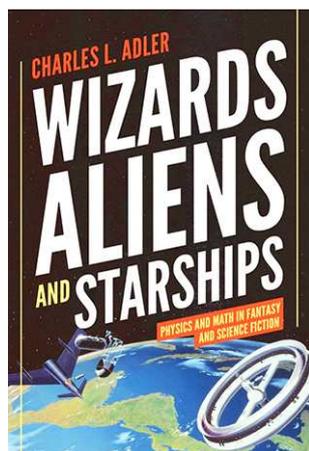
Also available in eBook (ISBN 978-1-400-84836-2)

Adler states that his purpose is to give a scientific critique of science fiction, focussing on the physics and mathematics. This book delivers on part of its promise: it does indeed discuss the physics and science of some science fiction and even fantasy standards. As far as maths goes, it is just the basic algebra needed to apply some physics formulae—occasionally the calculus behind it is discussed.

The choice of physics topics captures the main themes from science fiction, and throws in a few from fantasy as well, largely drawn from the Harry Potter books. While specifically staying away from movies and television, he does mention several, such as Star Trek and Star Wars.

The book starts with a discussion of the physics behind the Potter books, which gets the book rolling with its basic technique: check the energy requirements. Shape-changing is limited by conservation of mass (at least), dragons can't reasonably fly (metabolic rate versus power requirements), time travel causes no ends

of trouble, and so on. The chapter ends, oddly enough, with a long discussion about the dimness of candles compared to daylight and electric bulbs.



The book then settles down to science fiction proper. Topics covered are space travel, living and building in space, other habitable planets, aliens, the future of the Earth, humanity, and even the universe. This is good solid stuff, and the physics is discussed in enough detail to enjoy the borderlines of the plausible (space elevators and the Orion project) and the wildly unlikely flights of fancy (Ringworld).

The examples are drawn, almost exclusively, from the Golden Age of science fiction (1940–1960 or so, Asimov, Niven, Clarke, Heinlein, . . .) and a few books of the early twentieth century (mostly Stapledon). A couple of references to William Gibson and Greg Bear (and J.K. Rowling!) are the only exceptions. The focus on the Golden Age creates notable omissions, such as Frank Herbert and Iain Banks.

The book, and the extra problems available at the website, provide easy access to discussions of the rocket equation, twin paradox, wormholes and so on. Due to the format, these discussions will all require some effort to supplement, which is good. The equations are tested on real world numbers, often with much changing of units: tedious, but honest physics. The book skips from topic to topic quite quickly, which makes it much better to dip into than to read through. The editing is very uneven.

The main problem with this text for teaching physics is that the Golden Age of science fiction is now 50 years ago, and the examples almost certainly lack resonance for our students, though the topics still stand up. Even worse, these days much of the best science fiction is not even in books: movies, television and increasingly games like Halo and Destiny are where people pick up their science fiction.

Overall, I think this book is a handy reference for those interested in the science of science fiction. The good fun ideas are in the science fiction, with the science deployed to analyse it very workmanlike and a bit repetitive. The advantage is that the maths required and used is minimal, so that much of the text is accessible for first year physics students — nothing here for the maths students though.

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Origami Design Secrets

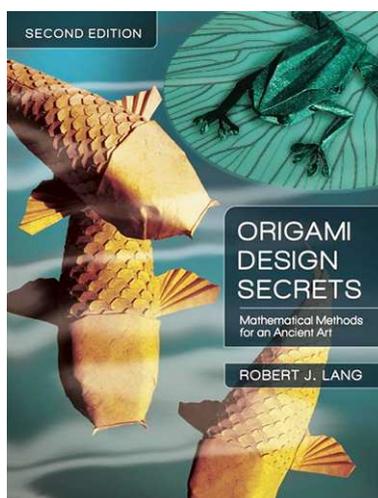
Robert J. Lang

A.K Peters, 2011, ISBN 978-1-568-81436-0

Introduction: Origami then and now

We may all recall folding little figures out of paper when we were younger: simple representations of animals and plants; maybe a few geometric shapes. The art of origami (the word comes from the Japanese for ‘paper folding’) has a long and honourable history in Japan (although it seems to have originated in China), mainly as a folk art. There are different traditions of origami, some allowing for cutting and pasting, or for non-square shapes of paper, but what we might call ‘classical origami’ uses a single uncut square.

Origami was long constrained by the fact that a square has four vertices. This meant that any figure crafted using origami principles would only have four ‘points’. A bird, for example, may have a head and tail and two legs, or a head and tail and two wings, but never head, tail, wings and legs together, as that would require six points. A quadruped—a horse for example—might be designed with only a single back leg, and some sort of rough squeezing about the head to indicate ears. Even with these constraints, folds could be used to great effect to indicate the shape of an animal, or some aspect of its behaviour. Precise verisimilitude was seen as unnecessary, and even by some as being inimical to the art.



However, concurrently with simple folding there was an undercurrent of a more serious formal approach, led in Japan by the great origami master Akira Yoshizawa (1911–2005). In his long career Yoshizawa created many thousands of models, from the simple to the enormously intricate, developing along the way techniques which made it possible to move beyond the simple four points to a greater representational exactness. Yoshizawa inspired other folders around the world, especially in the UK and the USA, and from the 1970s onwards there was a huge outpouring of origami models, with folders outdoing each other in the complexity of their designs.

Insects and other arthropods, long considered impossible as origami subjects due to their many legs, antennae, and other various pointy bits, have become favorites as models, and now there is a vast origami entomological bestiary, with models having the requisite number of legs (six), antennae (two), and any other points required by the species. Some insects, as well as legs and antennae, have (double) wings. Birds now have two legs (often with individual claws), sweeping tails, heads, and even in at least one example, wings with separate feathers. Quadrupeds have

all their legs, and tails, heads with ears, and anything else. Elephants have always been popular origami subjects, owing to the fact that their shape and bulk makes them easily recognizable. No modern folders worth their salt would dare to produce an elephant now with any less than four legs, a tail, a trunk, large flaps for ears, and tusks. And of course folders scorn the use of cutting, pasting, or using more than one sheet of paper. A model must start from a single square.

Possibly the most complex model in existence is a dragon by the Japanese artist Satoshi Kamiya, which includes a body covered with scales, four legs (all with long curving claws), a multi-horned head (with an open mouth and tongue), all folded from one (very large) uncut square, and which took several hundred hours to complete.

However, remarkable and astonishing as modern creations may be, origami is more than folding animals, and finding new ways to conjure points out of nowhere. It has become a serious applicable discipline, with applications including the folding and deployment of car air-bags; the folding and deployment of solar sails or telescopes on satellites; the folding of DNA molecules in order to provide transport of drugs to particular cells (for example cancerous cells, while leaving non-cancerous cells alone); the folding of arterial stents for easier delivery and use — the list goes on.

The third aspect of origami is its theory, to which its main contributor is Erik Demaine at MIT (who is the youngest ever professor to be appointed there), and who has raised the theory of origami to very respectable mathematical heights. In a mathematical sense origami theory may be considered a part of computational geometry, but like all theories, it has a character uniquely its own. Theorists concern themselves with considering, for example, if paper is folded, what constructions are possible? General angle trisections are almost trivial, as is the construction of non-euclidean lengths such as $\sqrt[3]{2}$. There is a collection of axioms: the Huzita–Hatori–Justin axioms, which precisely characterise the allowable folds: a point onto another point, a line onto a line, and so on. Quadratic, cubic and quartic equations with rational coefficients can be solved by origami constructions. There are also other questions: what are the conditions for folds to lie flat? What are the principles which allow a certain shape to be made? One of Demaine's earliest results is the 'fold and cut theorem', which says that any collection of straight line cuts in a piece of paper can be made by appropriately folding the paper first, and then cutting the folded paper with one straight cut.

Origami has evolved from its humble folk beginnings into a subtle art, an important area of applications, and with an impressive theory.

The book

Robert Lang has been at the forefront of both the art and its theory now for several decades, and is recognized internationally as a master folder. He has worked as an academic physicist, but now spends all his time working as an origami artist and consultant. He is one of the people involved in the car air-bags study mentioned above.

This is the second edition of a book initially published in 2003, and which differs from the first edition in having some more models, and one chapter removed. This chapter, which is available on Lang's website, contains only mathematics, and while its removal may be a disappointment to the readers of the AustMS *Gazette*, Lang claims that 'most people would not be interested in big, hairy equations'.

The book is primarily about the theory which makes the design of complex models possible. Clearly models of the complexity of an insect, or of Kamiya's dragon, can't be created by simply fiddlin' with a bit of paper and hoping for the best. Such models must be carefully designed first, and this book is an exploration of various methods which can be used to realize a particular shape starting with a single square. This design-oriented approach to origami is known in Japan as 'origami sekkei'; that is: 'technical origami'. This name is possibly a misnomer, as 'technical' sounds somewhat dry and pedantic, and yet the models produced are often of astonishing beauty.

The book starts and ends with elephants: first a page showing a 'herd' of 32 different origami elephants, from the very simple four-point variety (with a single rear leg) to some highly detailed modern examples, including one of Lang's own. The book ends with the folding instructions for Lang's elephant. Lang delightfully points out that in fact anatomical precision is not necessary for an origami model, and gives as an example a sort of 'zen' elephant in which the animal's likeness has been reduced to an extreme simplicity of form — with just one fold!

Although the book does not contain much formal mathematics in the sense of equations, theorems, and proofs (although there are some equations), the sense of the book is very mathematical, and much of terminology has mathematical antecedents.

The first few chapters discuss basic origami: the notation and symbols used for diagrams (developed by Yoshizawa, extended by other folders, and now universal), and the use of simple 'bases'. In origami, a 'base' is the result of several folds into a form which can be used to create different models. Bases are named after their most famous models, thus the 'bird base', 'frog base' and so on. Bases were fundamental to much classical origami, and they have a new lease of life in this book. Chapter 4 uses the standard bases in highly non-standard ways to produce some remarkable models, for which full folding instructions are given: a stealth fighter, a snail (with a lovely shell), a valentine (heart with arrow through it), a ruby-throated hummingbird (in which the colours on the different sides of the paper are cleverly used), and a sitting baby.

Chapters 5–14 introduce the techniques of origami sekkei: Splitting Points, Grafting, Pattern Grafting, Tiling, Circle Packing, Molecules, Tree Theory, Box Pleating, Uniaxial Box Pleating, and Polygon Packing. The book finishes with Chapter 15: Hybrid Bases, in which Lang discusses the design of models which don't have masses of points but which may include large flat areas, such as butterfly wings. Each chapter begins with an essay describing the technique and its uses, then gives some simple examples (with numerous diagrams), showing how it might be used in practice, and then finishes off with the folding instructions for various models illustrating the technique. As to be expected, in this book the models are

mostly complex, although there are a few which are within the purview of the intermediate folder. One such model, in Chapter 9, Circle Packing — is an emu!

As example chapters, take Chapters 9 and 10: Circle Packing, and Molecules. Points in a model, if not folded from the square's corners, must come from the middle somewhere. When the model is unfolded, the resulting pattern of creases on the paper, known not surprisingly as a 'crease pattern', will show a local radial symmetry about the positions of the model's points. A model may therefore be designed by first placing circles of the right size on the square: the centres of the circles will end up as points in the model; the lengths of radii indicating the lengths of those points. For efficiency it will be necessary to align the circles so that they are mutually tangential; this is in fact a difficult computational procedure (see the Addendum below), and Lang provides a neat method of doing this with 'jigs' (cardboard circles with upturned drawing pins through their centres), and two L-shaped rulers to make the square. It would be interesting to see if Lang's jig method could be realized with dynamic geometry software or a computer algebra system, but this seems to be an area as yet unexplored. The emu is given as an example of offsetting the centre of the bird base so as to obtain a longer neck.

Chapter 10 again has lots of circles — and in passing one of the book's many excellent aspects is the manner on which chapters build on previous material — but in this chapter Lang looks at how folds can be made so that their edges line up. With a triangle this is trivial, for a convex quadrilateral Lang shows that such a fold is possible if and only if the sums of opposite sides are equal. Although this could be stated and proved as a formal theorem, it is not given as such, as this text is not designed for mathematicians, but for origami artists and designers. Such elements of a fold are called 'molecules' (and so named by another folder, the Japanese Toshiyuki Meguro, who is a biochemist). Lang claims that: 'By enumerating and identifying the molecules of origami, we will develop the building blocks of origami life'. Lang then enumerates different molecules, and describes how molecules can be used in that part of a design between the circles: places Lang refers to as 'rivers'. The combination of circles and rivers can be used to create extraordinarily complex models, in particular arthropods, and one of this chapter's two models is a silverfish, with six legs, two antennae, two palps (appendages next to the antennae which are used for tasting and manipulating food), three long tail segments (which are formally called 'cerci') plus two other rear appendages, as well as a tapering scaly body. That's fifteen points!

Some of the book's most complex models are not given as a sequence of folding instructions, but as crease patterns, which illustrate the design elements introduced earlier in the book. These models are all insects and arachnids, and as you would expect, contain points galore. One insect, a 'Euthysanius Beetle', as well as having two tarsal claws on each leg, has combed antennae. The result is a tour-de-force of origami. (And so are all the others.)

Who is this book for?

Although this is not a formal mathematics book as such, it has immense mathematical appeal. The intricacy of the diagrams and models alone, along with the careful theory which underlies their design, make for an enticing book. Even if you never make any of the models, you can admire them. If you have played around with some simple origami and would like to advance your skills—both as a folder and designer—this book is a cornucopia of good things. You may not necessarily become a virtuoso folder like Lang, any more than reading through a book of music theory will turn you into Mozart, but you will certainly advance both your folding and design skills, and maybe create some lovely models of your own. And in doing so, you may be quietly entering the enticing world that is origami sekkei.

Addendum

Because this book is not a mathematics text, it does not mention a remarkable result proved in 2010 by Lang, Demaine, and Sándor Fekete: that the design of a crease pattern, in particular the placing of circles into a square, is NP-hard. So not only is origami difficult to do, it is computationally intractable!

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Circular statistics in R

Arthur Pewsey, Markus Neuhäuser and Graeme D. Ruxton

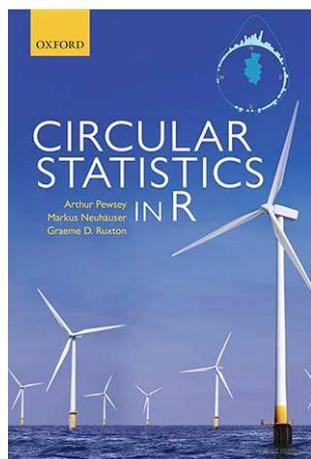
Oxford University Press, 2013, ISBN 978-0-19-967113-7

Directional, and in particular circular, statistics date back to Fisher and von Mises who were the first ones to develop ideas for such data. Applications of such methods can be found especially in ecology, geology and seismology, or earth sciences as a whole.

I was glad to read a book in circular statistics, which focuses on real data examples and it contains R functions. N.I. Fisher's *Statistical Analysis of Circular Data*, E. Batschelet's *Circular Statistics in Biology* and *Topics in Circular Statistics* by S.R. Jammalamadaka and A. SenGupta are three other books specifically focused on circular statistics but they either lack or have very few examples with statistical software within the book.

The book is suitable for postgraduate students in statistics or mathematics or related fields (with knowledge in statistics). This book is a useful companion for Fisher's book (*Statistical Analysis of Circular Data*). It is not a very introductory book for circular statistics though (it does not start from scratch, or contain a lot of theory to begin with), but on the other hand the fact that it's not very technical

and it contains a lot of R functions is what makes it suitable for practitioners as well. So I would say there is a good balance between mathematics and R functions.



The first chapter contains a long introduction, which is nicely written and informative. The next two chapters describe graphs and descriptive statistics for circular data, with not many technicalities. Chapter 4 contains a list of circular distributions but again with not many details, even though this could be a drawback, since in some cases the mean direction and the circular variance are not provided. The remaining chapters go deeper into the field but again with not too many technicalities: hypothesis testing, model fitting, correlation and regression are discussed.

In most cases, an R library is used extensively, but in many cases the authors provide some of their own R functions. This is a good feature of the book. The interested reader may try to write his own functions and compare the results with the available ones. The practitioner who wants to analyse some circular data without having to write his own functions or search on the internet for functions, is good to go. The authors have made all their programs available online. The data however are not, and perhaps they could upload them also, at some point. To be honest it took me some time to understand how the front page relates to circular data, wind mills generating electric energy. Wind speed is measured in degrees, so here you are, circular data. A drawback of the book I found is that the fonts are too small. My sight is not the best, so I needed some time for my eyes to get used to these fonts. The book consists of only 172 pages in total and this is an attractive feature and the examples used to illustrate the analyses come from real life. It's like a short manual for circular data analysis. I would like to see a similar book for spherical data as well.

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Linear Algebra: Step by Step

Kuldeep Singh

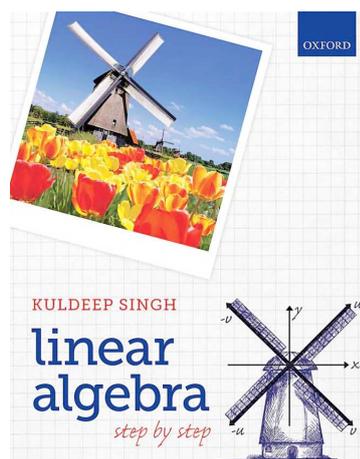
Oxford University Press, 2013, ISBN 978-0-199-65444-4 (paperback)

Linear algebra is an important subject not only to mathematicians but also to an ever widening range of disciplines. Time and time again authors note, typically in their text's preface, that in the old days only mathematics and physics majors would be in a foundation linear algebra course. Now, however, due to the wonders of the computing age, various aspects of linear algebra have found application in

diverse areas which in turn means that the neophytes of other departments can be seen attending linear algebra 101.

The effect of the broader audience is reinforced by the trend of lower levels of mathematical preparedness students have after finishing high school. It is with today's 'average' student that Kuldeep Singh, senior lecturer in mathematics at the University of Hertfordshire, has written his undergraduate text *Linear Algebra: Step by Step* (LASBS). Singh's goal is to assume only a limited mathematical background while still covering difficult material and to present it in such a way that students are able to learn on their own without the constant support of lecturer or tutor.

Like many texts in linear algebra, this one begins with systems of linear equations and uses that as an obvious vehicle to introduce matrices and Gaussian elimination. Matrix algebra, elementary matrices, and inverses follow. Vectors in \mathbb{R}^2 and \mathbb{R}^3 form the topic of the next chapter together with linear independence, basis and spanning sets. General vector spaces are delayed until the third chapter where many early concepts are revisited. Inner products are covered in Chapter 4 followed by linear transformations. The text is rounded out with chapters on determinants and eigenvalues/vectors finished with sections on LU decomposition and singular value decomposition respectively. Between chapters is an interview with an academic or industrial scientist who uses linear algebra in their research. Scattered throughout the text are biographical notes on famous mathematicians who had an impact on linear algebra.



LASBS is an aesthetically pleasing and relatively cheap textbook (under \$55) of 608 pages. It is a much more inviting read than Singh's earlier *Engineering Mathematics Through Applications* and the more challenging content isn't infected with a rash of Greek symbols or dense prose which can put off anyone let alone Singh's target audience of so-called 'average' students. The usual exercise style of questions end sections with brief solutions contained in the book. (The student can go on-line to the LASBS website for full worked solutions and additional material.) More challenging questions are taken from the exam papers from various universities around the world with the idea of extending the student and building their confidence. The idea being that if they can solve an exam question from 'insert name of famous institution here' then they are making progress.

Importantly, throughout the text questions are posed to the student in blue font which are then promptly answered. While many authors pose the question 'why?' after some statement to encourage thought on the topic at hand it appears to me that Singh is more concerned with making questions like 'What does this definition mean?', 'How could we prove this statement?', 'Can the statement be generalised?', 'When is this valid?' common place and showing the 'average' student that they

should be asking these questions and how they might answer the questions for themselves. Questioning the content is important but quite a rare activity amongst students in my experience who seem to focus more on calculation and getting an ‘answer’.

With LASBS I believe Singh succeeds starting off with a low expected background and building in abstract topics such as vector spaces and generalised inner products. I’m somewhat sceptical of Singh’s claim that students will be able to take all this in on their own. Perhaps a better than average student with an interest in mathematics could but the target students, particularly if maths is not their primary focus, are going to be less inclined to read on their own and more inclined to jumping straight to this week’s assignment questions. Getting such students reading and thinking about the text in blue font is where the instructors come in.

There are no chapters on numerical methods or on common packages such as MATLAB which some may see as bad. I’m inclined to see it as a positive as LASBS keeps the focus on understanding the mathematics rather than calculating. Depending on the scope of your course or time constraints you may wish to drop the harder material—at least the first time round—and LASBS allows this by having chapters mainly self-contained.

If you don’t already have your own set of notes or a favourite text to set students LASBS is definitely worth considering, particularly if you are wanting to ease students towards more abstract topics.

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