Lift-Off Fellowship report
Geodesics: a natural breeding ground for geometrical problems

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For centuries geodesics have been the breeding ground for many mathematical problems. In our research, we have exploited once more their unlimited generosity in several different ways [1], [2] but here, to keep ourselves within the limits of a short report, we mention just two of them.

Let Σ be a smooth surface in 3-dimensional Euclidean space. Allow a particle on Σ with some initial position and velocity to move under no forces except the minimum required to keep it constrained to Σ. The trajectory it follows (both forwards and backwards in time) is called a *geodesic*. For example, the geodesics of a sphere are its great circles.

Lorentzian manifolds are the natural ambient spaces for the study of relativity. Geodesic motion can be extended to Lorentzian manifolds in equivalent different ways. For instance, they are constant speed curves which are critical for the curve energy. But also, they can be seen as trajectories of particles traveling with vanishing acceleration.

Moreover, geodesics provide solutions for other classical variational problems as, for example, the elastica problem. According to D. Bernoulli's model (1742) an elastica is a minimizer of the bending energy of the curve and geodesics are the obvious examples since they are absolute minimizers. Elasticae in ambient spaces of constant curvature have been well studied since the Euler solution for the plane elastica problem (1744). However, in non-constant curvature ambient spaces, the elastica problem is much more difficult to deal with and very little is known about it.

We have studied a generalized elastica problem for clamped curves which are constrained to lie on a surface of a 3-dimensional space with constant curvature (surface constrained problem). We were particularly interested in the total curvature energy, for which the curves are related to Plyushchay's model for relativistic particles. We observe that a geodesic of a surface need not be a critical curve for the surface constrained problem and then we find the differential equation to be satisfied for a critical geodesic. In order to find explicit solutions, we construct surfaces locally foliated by geodesics satisfying such a differential equation.

Suppose Σ is locally parameterised as $\Sigma(u, v)$. If, for each u, the curve $v \mapsto \Sigma(u, v)$ is a geodesic, we say that the family of these curves forms a *foliation by geodesics*.

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For example, if you remove the north and south pole of a sphere, the lines of longitude form a foliation of the resulting surface by geodesics (see Figure 1). All surfaces can be locally foliated by geodesics.

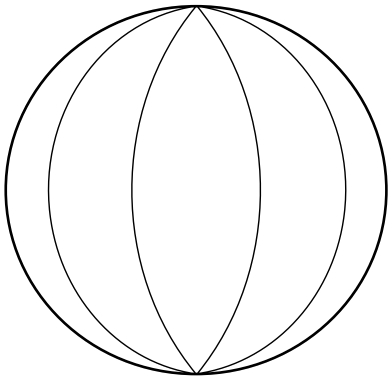

Figure 1. Foliation of a surface by geodesics.

Returning to our problem, one may prove that if a geodesic is critical for the surface constrained relativistic particle model then it must have constant torsion. So we wish to construct examples of surfaces locally foliated by geodesics with constant torsion (we measure torsion in the ambient 3-dimensional space). The partial differential equations which arise from this problem have many solutions, but giving explicit examples is not so simple. We restricted our attention to the surfaces which can be described with a geometrically simple construction.

In Euclidean space, curves with zero torsion are planar. So, one special case consists of surfaces in Euclidean space that have a foliation by planar geodesics (an example is the foliation by lines of longitude in Figure 1). All such surfaces can be locally described by the following construction. Let δ be a curve in 3-dimensional Euclidean space and let γ be a curve in the Euclidean plane. At any point p along δ, we can copy γ somewhere into the normal plane to δ at p. If we choose one copy for each p in a smooth way, they will come together to form a surface Σ, except possibly at points where δ curves so sharply that the copies of γ start to overlap. Everywhere else, the curves form a foliation of Σ by planar curves, although they are not necessarily geodesics.

The trick now is to choose the family of copies in a way that twists as little as possible — then the copies of γ will be geodesics in Σ. This construction can be given in terms of the torsion and the normal and binormal vectors of δ. See Figure 2. The copies of γ form a foliation of Σ by planar geodesics.

We found related results for surfaces in the 3-sphere and hyperbolic 3-space, and some special cases for nonzero constant torsion. We also give some results concerning the elastica constrained problem [1].

In [2] we study another question, related to the characterization of geodesics as trajectories of vanishing acceleration. Amongst other results, we prove that unit speed curves in Riemannian manifolds with vanishing higher order accelerations

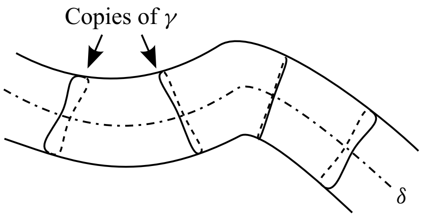

Figure 2. A construction in Euclidean space.

have to be geodesic, as expected. Surprisingly enough, we show that this is not true in Lorentzian spaces of arbitrary dimension, but it remains valid in Lorentzian surfaces. This is a neat difference between Riemannian and Lorentzian behaviour.

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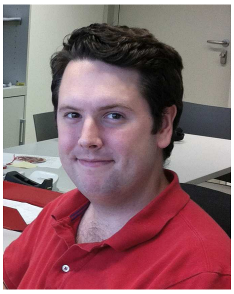
References

[1] Garay, O.J. and Pauley, M. (2013). Critical curves for a Santaló problem in 3-space forms. *J. Math. Anal. Appl.* **398**, 80–99.

[2] Garay, O.J. and Pauley, M. Non-geodesic particle trajectories with vanishing higher accelerations. To appear in *J. Geom. Phys.*



Michael did his undergraduate and PhD studies in pure mathematics at the University of Western Australia. He earned his PhD in 2011 for his thesis, *Cubics, Curvature and Asymptotics*, supervised by Winthrop Professor Lyle Noakes. During the Lift-Off Fellowship, Michael visited Prof Óscar Garay at the University of the Basque Country in Spain. He has since worked as a consultant in discrete event simulation, and is now a postdoc at Johannes Kepler University in Austria, where his work is related to isogeometric analysis.