Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 32. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, School of Science, Information Technology & Engineering, University of Ballarat, PO Box 663, Ballarat, Vic. 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 32 is 1 July 2013. The solutions to Puzzle Corner 32 will appear in Puzzle Corner 34 in the September 2013 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Telescoping product

Let n be an integer greater than 1. Simplify

$$\frac{2^3-1}{2^3+1}\times\frac{3^3-1}{3^3+1}\times\cdots\times\frac{n^3-1}{n^3+1}.$$

Tangent intersections

Let Γ_1 and Γ_2 be two non-overlapping circles with centres O_1 and O_2 respectively. From O_1, draw the two tangents to Γ_2 and let them intersect Γ_1 at points A and B.

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This puzzle corner is also featured on the Mathematics of Planet Earth Australia website http://mathsofplanetearth.org.au/

Similarly, from O_2, draw the two tangents to Γ_1 and let them intersect Γ_2 points by C and D.

Prove that $AB = CD$.

Colour coordination

Submitted by Joe Kupka

I need to hang 20 garments on a clothes line. Each garment requires two pegs. I have 20 green and 20 red pegs. I choose pegs at random. On average, how many garments will have pegs of the same colour?

Team tactics 2

In a game show, there are three girls, each wearing a blue or a red hat. Each girl can only see the hats of the other two but not her own. Without any communication between themselves, each girl has to choose a real number and whisper it to the host. At the end, the host will add up the numbers chosen by girls wearing red hats, then subtract the numbers chosen by girls wearing blue hats. The girls win if the final answer is positive.

Before the show, the girls try to devise a strategy to maximise their probability of winning.

(i) What is the maximum probability of winning?

(ii) If the girls were only allowed to choose from $\{-1, 0, 1\}$, what is the maximum probability of winning?

Bonus: If there are seven girls instead of three, and each girl can see the hats of the other six but not her own, how do the answers change?

A sequence of sequences

Let $S_1, S_2, \ldots$ be finite sequences of positive integers defined in the following way. Set $S_1 = (1)$. For $n > 1$, if $S_{n-1} = (x_1, \ldots, x_m)$ then

$$S_n = (1, 2, \ldots, x_1, 1, 2, \ldots, x_2, \ldots, 1, 2, \ldots, x_m, n).$$

For example, the next few sequences are $S_2 = (1, 2)$, $S_3 = (1, 1, 2, 3)$ and $S_4 = (1, 1, 1, 2, 1, 2, 3, 4)$.

Prove that in the sequence S_n where $n > 1$, the kth term from the left is 1 if and only if the kth term from the right is not 1. (*Hint:* Pascal's triangle.)

Solutions to Puzzle Corner 30

Many thanks to everyone who submitted. The $50 book voucher for the best submission to Puzzle Corner 30 is awarded to Dave Johnson. Congratulations!

Peculiar pace

Jodie jogged for 25 minutes. In any 10-minute period, her average speed was 18 kilometres per hour. How far did she run?

Solution by Darren O'Shaughnessy: Divide the 25 minutes up into 5 sessions of 5 minutes. Since Jodie jogged exactly 6 km in the first 4 sessions and no more than 3 km in the last session, the answer has to be between 6 km and 9 km. If Jodie is a robot, then any value between 6 and 9 is possible. For example, to achieve $(6+x)$ km, where $0 \leq x \leq 3$, she could jog at $12x$ km/h during sessions 1, 3 and 5, and jog at $(36 - 12x)$ km/h during sessions 2 and 4.

However, assuming Jodie is human, she would likely be constrained by physiological limits. The record speed over five minutes by a woman is approximately 22.2 km/h. If Jodie could manage that three times in 25 minutes, the upper limit would be $3 \times 22.2/12 + 2 \times (36 - 22.2)/12 = 7.85$ km. But it is likely to be substantially less due to the cumulative fatigue from three bursts of world-record pace.

Comment by Joe Kupka: If the '10-minute period' does not have to be a connected interval, then the answer would be 7.5 km.

Rolling roadblocks

There are 10 cars on an infinitely long, single-lane, one-way road, all travelling at different speeds. When any car catches up to a slower car, it slows down and stays just behind the slower car without overtaking. Eventually, the cars form a number of separate blocks. On average, how many blocks do you expect to see?

Solution by Pratik Poddar: We will solve the general problem of n cars. Car i (ith car from the front) will be at the start of a block if and only if it is slower than all of the cars in front. Out of the first i cars, each of them has equal chance of being the slowest. Hence the chance of car i being the slowest of the first i cars is $\frac{1}{i}$.

Now the number of blocks formed is also the number of cars being at the start of a block. By linearity of expectations,

$$\begin{aligned}\text{Expected number of blocks} &= \sum_{i=1}^{n} P(\text{Car } i \text{ is at the start of a block}) \\ &= \sum_{i=1}^{n} \frac{1}{i}.\end{aligned}$$

So the required answer is $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{10}$.

Polynomial parity

(i) *Let P and Q be complex polynomials with no common factors. Suppose the rational function P/Q is an even function. Prove that P and Q are both even functions.*

(ii) *Let P, Q and R be complex polynomials. Suppose PQ, PR and QR are all even functions. Prove that either P, Q and R are all even functions, or they are all odd functions.*

Solution by Dave Johnson:

(i) In P, we may separate the odd and even degree terms by writing $P(z) = A(z^2)z + B(z^2)$ for polynomials A and B. Similarly write $Q(z) = C(z^2)z + D(z^2)$. Using the shorthand $A = A(z^2)$ etc., we have

$$\begin{aligned} & P(z)/Q(z) = P(-z)/Q(-z) \\ \implies \quad & (Az+B)(-Cz+D) = (-Az+B)(Cz+D) \\ \implies \quad & AD - BC = -AD + BC \\ \implies \quad & AD = BC. \end{aligned}$$

Thus

$$DP = ADz + BD = BCz + BD = BQ.$$

Since P and Q have no common factors, the polynomial $Q(z)$ must be a factor of $D(z^2)$. But from $Q(z) = C(z^2)z + D(z^2)$, the degree of $Q(z)$ must be at least as big as the degree of $D(z^2)$. Hence $Q(z)$ and $D(z^2)$ must have equal degrees and $Q(z) = kD(z^2)$ for some $k \in \mathbb{C}$. Therefore Q is an even function. Since P/Q is even, it follows immediately that P is an even function as well.

(ii) Using the notation from part (i), we have

$$\begin{aligned} & P(z)Q(z) = P(-z)Q(-z) \\ \implies \quad & (Az+B)(Cz+D) = (-Az+B)(-Cz+D) \\ \implies \quad & AD + BC = -AD - BC \\ \implies \quad & AD = -BC. \end{aligned}$$

Thus

$$D(z^2)P(z) = ADz + BD = -BCz + BD = B(z^2)Q(-z).$$

If we write $R(z) = E(z^2)z + F(z^2)$, then by the same argument,

$$F(z^2)Q(z) = D(z^2)R(-z), \qquad B(z^2)R(z) = F(z^2)P(-z).$$

Hence

$$\begin{aligned} F(z^2)D(z^2)P(z) &= F(z^2)B(z^2)Q(-z) \\ &= B(z^2)D(z^2)R(z) \\ &= F(z^2)D(z^2)P(-z). \end{aligned}$$

Now there are three cases. Keep in mind that PQ, QR and RP are all even functions.

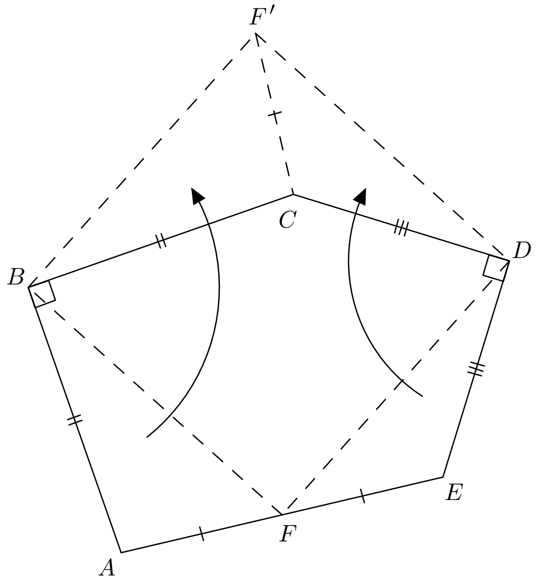
- If $D = 0$, then $Q(z) = C(z^2)z + D(z^2) = C(z^2)z$ is an odd function. Thus P and R are also odd functions.
- If $F = 0$, then $R(z) = E(z^2)z + F(z^2) = E(z^2)z$ is an odd function. Thus P and Q are also odd functions.
- Finally, if F and D are both non-zero, then $P(z) = P(-z)$ must be an even function. Thus Q and R must be even functions as well.

Squaring a pentagon

Let $ABCDE$ be a pentagon with $AB = BC$, $CD = DE$ and $\angle B = \angle D = 90°$. Can you cut the pentagon into three pieces and then rearrange them to form a square?

Solution by Jensen Lai: Let F be the midpoint of AE. Cut along lines BF and DF to form three pieces.

Since $AB = CB$, we can rotate $\triangle ABF$ about B until A coincides with C. Similarly, since $ED = CD$, we can rotate $\triangle DEF$ about D until E also coincides with C. Since angles B and D are right angles, angles A, C and E must sum to 360°. Hence after the two rotations, the line AF should coincide with the line EF. Furthermore, since $AF = EF$, the corners of both rotated triangles will meet at F'.



Now $\angle F'DF = \angle CDF + \angle F'DC = \angle CDF + \angle FDE = 90°$. Similarly $\angle F'BF = 90°$. Finally, since $DF = DF'$ and $BF = BF'$, the quadrilateral $BFDF'$ has to be a square.

Team tactics

In a game show, a team of n girls is standing in a circle. When the game starts, either a blue hat or a red hat is placed on the head of each girl. Due to the set-up of the stage, each girl can only see the hats of the two adjacent girls, but not her own hat nor the hat of anyone else. Without any communication, the girls have to simultaneously guess the colour of their own hats. The team wins if and only if everyone guesses correctly.

Before the show, the girls try to devise a strategy to maximise their probability of winning. What is the maximum probability of winning if

(i) $n = 3$?
(ii) $n = 4$?
(iii) $n = 5$?

Solution by Ross Atkins (problem submitter): Label the girls A, B, C, etc. going around the circle.

(i) For $n = 3$, use the following strategy:
- If the two adjacent hats are the same, guess blue;
- If the two adjacent hats are different, guess red.

The following colour combinations will result in a win

$$BBB, \quad BRR, \quad RBR, \quad RRB.$$

This amounts to $4/2^3 = 50\%$ chance of winning.
Since regardless of the strategy, girl A only has a 50% chance of guessing correctly. Therefore, the team cannot do better than 50%.

(ii) For $n = 4$, use the following strategy:
- A and B guess each other's hat colours;
- C and D guess each other's hat colours;

The following colour combinations will result in a win

$$BBBB, \quad BBRR, \quad RRBB, \quad RRRR.$$

This amounts to $4/2^4 = 25\%$ chance of winning.
Both girls A and C will have to base their guesses on the same available information: the hat colours of girls B and D. Also, since they cannot see each other, their chance of success are independent. Now each of A and C only has a 50% chance of guessing correctly. Therefore the team cannot do better than 25%.

(iii) For $n = 5$, use the following strategy:
- If the two adjacent hats are the same, guess the opposite colour;
- If the two adjacent hats are different, guess Red.

The following colour combinations will result in a win

$$RBRBR, \quad BRBRR, \quad RBRRB, \quad BRRBR, \quad RRBRB.$$

This amounts to $5/2^5 = 15.625\%$ chance of winning.

To show that the girls cannot do better, we argue by contradiction. Assume that there is a strategy which results in six or more different winning colour combinations out of 32. Consider five of these combinations $c_1, \ldots, c_5$.

Take a pair of girls which are not adjacent, say (A, C). There are four possible colour pairs for (A, C), so there must be two combinations containing the same colours for (A, C). Since girl B must guess based solely on the hat colours of A and C, the hat colour of B is forced. So there exist two combinations containing the same colours for the triple (A, B, C). The same can be said about the triples $(B, C, D), (C, D, E), (D, E, A)$ and (E, A, B).

Consider a graph with the combinations $c_1, \ldots, c_5$ as vertices. For each of the five triples $(A, B, C), (B, C, D), \ldots, (E, A, B)$, place an edge between two combinations if they have the same colour for that triple. By the earlier argument, at least one edge of each type must be present. Now there are a few useful properties.

1. By the earlier argument, *there is at least one edge of each type and at least five edges in total.*
2. *No two combinations can agree on four of the five colours.* Otherwise the last colour is forced by the neighbours and the two combinations are identical.
3. *There cannot be more than one edge between two combinations.* Otherwise the two combinations will be agreeing on at least four colours, violating rule 2.
4. *The combination connected to the edge (A, B, C) cannot connect to the edge (B, C, D).* For example, if we have

$$c_1 \overset{(A,B,C)}{\longleftrightarrow} c_2 \overset{(B,C,D)}{\longleftrightarrow} c_3,$$

   it follows from rule 2 that the colour of E in both c_1 and c_3 must be opposite to c_2. Hence c_1 and c_3 have the same colours for B, C and E. But then the colours of A and D are forced by their neighbours and c_1 and c_3 are actually identical. Note that this rule may be applied to other triples too, up to cyclic permutations.

From rules 1, 3 and 4, it is clear that each combination is connected to exactly two other combinations, forming a cycle. The graph must be, up to the relabelling of combinations,

$$c_1 \overset{(A,B,C)}{\longleftrightarrow} c_3 \overset{(C,D,E)}{\longleftrightarrow} c_5 \overset{(E,A,B)}{\longleftrightarrow} c_2 \overset{(B,C,D)}{\longleftrightarrow} c_4 \overset{(D,E,A)}{\longleftrightarrow} c_1.$$

Now if there is a 6th winning combination c_6, we may apply the same arguments to c_1, c_2, c_3, c_4, c_6. So c_6 is forced to take the place of c_5 in the graph. By the definition of the edges, we see that c_6 must be identical to c_5, which is a contradiction.

Therefore there are at most five winning combinations out of 32, and the team cannot do better than $5/32 = 15.625\%$.

Note: We did not explicitly investigate randomised strategies in the solution. The reasoning is as follows: In this problem, the winning probability of any randomised strategy can be written as a weighted sum of the winning probabilities of deterministic strategies. So in order to maximise the winning probability, it suffices to only check the deterministic strategies.

Furthermore, the girls may not be allowed access to any randomisation devices (coins, dice, etc.) during the game show anyway!

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Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.