Tennis anyone?
Courtside combinatorial conundrums

Scott Sciffer*

1. Introduction

During a break in play the conversation drifted towards what we all did for a crust. ‘Yes, but other than teach, what does a mathematician actually do?’

We’ve all been there. An awkward silence ensued. Where could I start? Most other professions can be explained simply to the layman; indeed most need no such explanation. But the leap from arithmetic to Banach space theory is too big a gap to span simply. The awkward silence continued. To ease my discomfiture someone changed the subject.

‘Has anyone seen the draw for the next comp? Some teams play each other several times, but others don’t meet at all.’

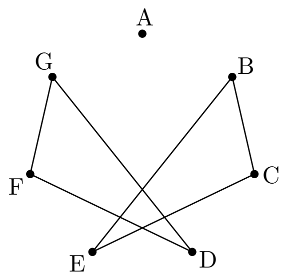
It was a seemingly harmless remark, but I saw it for what it was: an opportunity.

‘Here, give me a look at that’, I said.

2. The problem with ‘threesomes’

The format of our competition pits three teams against each other every round. If teams A, B and C are drawn to play each other, then in the same afternoon they play the three possible match-ups: A vs B, A vs C and B vs C. If the number of teams in the whole competition is a multiple of three, every team plays every round; otherwise a ‘bye’ is incorporated in the draw. For example, with seven teams labeled A through G, a draw might look like:

Round	Court 1	Court 2	Bye
1	B vs C vs E	D vs F vs G	A
2	C vs D vs F	D vs G vs A	B
3	D vs E vs G	D vs A vs B	C
4	E vs F vs A	D vs B vs C	D
5	F vs G vs B	D vs C vs D	E
6	G vs A vs C	D vs D vs E	F
7	A vs B vs D	D vs E vs F	G



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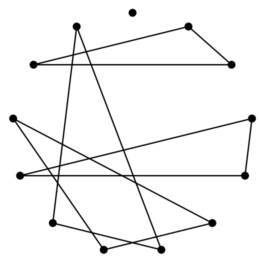
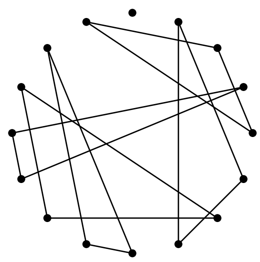
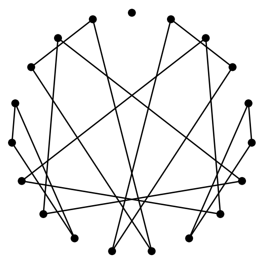
This is an example of a *cyclic* draw as it is generated by rotations of the adjacent diagram. The diagrammatic representation is more succinct and can easily be seen to give the desired result; even by a bunch of middle-aged tennis players! Notice that each triangle joins teams which are 1-apart, 2-apart and 3-apart around the edge of the circle. Therefore after seven rotations we see that all teams will have been paired against each other exactly twice, and each team will have had the bye exactly once. A 'perfect' draw. Perfect cyclic draws for competitions involving 13, 16 and 19 teams are shown below.

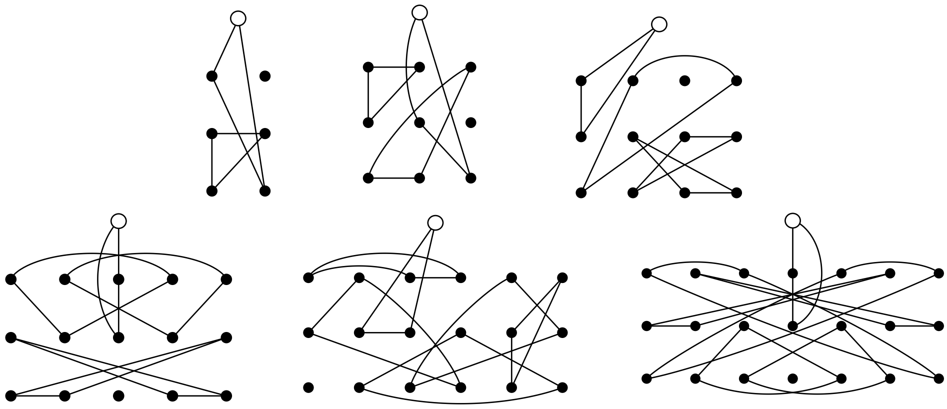
I had just finished explaining and drawing these diagrams on the back of an envelope, and was feeling satisfied that I had upheld the honour of my profession when someone pointed out that there were 10 teams in the next competition. This was a setback, because:

Theorem. There is no 'perfect' cyclic draw for 10 teams.

Proof. Suppose there was. Then the associated diagram would have to consist of three triangles and the 'distance'-apart of the nine edges would have to be $\{1, 1, 2, 2, 3, 3, 4, 4, 5\}$, which adds up to 25, *an odd number*. However, any individual triangle you draw is either a 'short triangle' whose longest edge is the sum of the two shorter edges, or is a 'long' triangle whose edges add up to 10. Either way, the total for any one triangle is *an even number*, so it is impossible to draw three triangles whose edges total 25. □

I was going to mention that this argument actually applies to any number of teams congruent to 10 mod 12, but thought better of it. To impress the efficacy of mathematics on my audience it was more important to solve the immediate problem. I opted for a different group action.

Consider the six diagrams below, representing 'perfect' draws for competitions involving 7, 10, 13, 16, 19 and 22 teams respectively. Cycle each diagram horizontally and vertically, keeping the white circle fixed, to generate the different rounds of the draw. When that has been done, all teams except the one represented by the white circle have had their bye. For the last round of the competition the teams in vertical alignment play each other while the white one has the week off. (To see that these diagrams actually work it may help to consider all the different line segments which can be drawn modulo the group action.)



At this point the back of the envelop was full, and it was time to head back out on court, much to everyone's relief.

[Of course all of this is well known. The problem posed here is a variation of the famous Kirkman's Schoolgirl Problem called a *near-resolvable design*. In this case we were seeking an $NRB(t, 3, 2)$, and these are known to exist for all $t \equiv 1 \bmod 3$, [1, p. 128].]

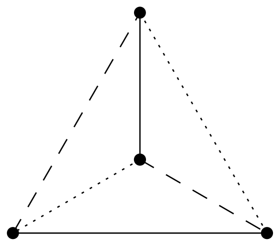
3. The problem with 'foursomes'

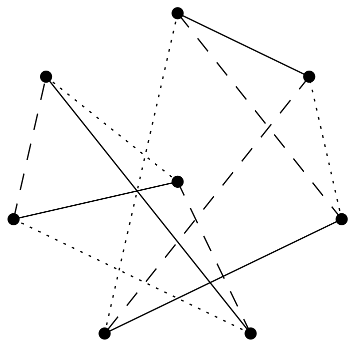
That was Saturday afternoon. Wednesday night is another competition, at a different venue, with a different format. In fact it is *two* competitions in one! Teams consist of two players of different ability levels, say an A-grader and a B-grader. In each round of the competition, a pair of teams are pitted against each other, with the results contributing to the *team* competition. In addition, players of the same ability level on one court are pitted against the corresponding players on the adjacent court, with the results contributing to an *individual* competition (either A-grade or B-grade). To understand this system, consider four teams assigned to adjacent courts. The matches to be played are:

Court 1	**Court 2**
1A vs 2A	3A vs 4A
1B vs 2B	3B vs 4B
1A&1B vs 2A&2B	3A&3B vs 4A&4B
1A&2A vs 3A&4A	1B&2B vs 3B&4B

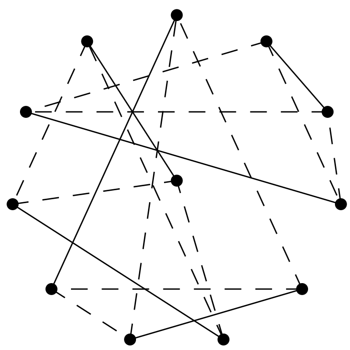
The first three sets contribute to the team pointscore, while only sets played against opponents of the same ability level contribute to the individual pointscore. Hence the singles matches contribute to both competitions. (It may seem a bizarre system, but in fact I highly recommend it. Each set poses different challenges.)

In an ideal draw, every team would play every other team once. This is easily arranged. However the way we arrange these team matches on adjacent courts *induces* the draw for the 'cross-over' doubles matches. Ideally we would like each player to play against each potential opponent exactly twice in those matches. Consider cases where the number of teams $t \equiv 0 \mod 4$, so no bye is required. Draws for four or eight teams can be generated by rotations of the following diagrams around their central points.





The solid lines represent the team matches while the dashed and dotted lines show the induced 'cross-over' doubles matches. Note that in these cases, upon rotation, each line (solid, dashed or dotted) gives rise to a factoring of the complete graph, K_4 or K_8 respectively. However, we might suspect that as the number of teams increases the need to simultaneously satisfy the requirements of both the team and individual competitions will become more difficult. And so it proves. For $t = 12$ it is not possible to achieve such a simultaneous factoring; at least not in this cyclic fashion. However, it is enough for our purpose that the dashed and dotted lines factor the 2-fold complete graph. Consider:



Here the solid lines, representing the team matches, lead to a factoring of K_{12} while the dashed lines, representing the induced 'cross-over' doubles matches, give rise to a factoring of $2K_{12}$. Unfortunately that's as far as we can go. For $t = 16$ no (cyclic) ideal draw is possible.

As you can imagine, it is not often the number of teams in a competition is a neat multiple of four. The current competition has 10 teams, requiring two teams to have a bye each week. What's the best draw you can devise?

References

[1] Colbourn, C.J. and Dinitz, J.H. (eds) (2007). *Handbook of Combinatorial Designs*, 2nd edn. Chapman & Hall/CRC, Boca Raton, FL.

Scott Sciffer is a lecturer in Mathematics within the Open Foundation unit at the University of Newcastle. As a mathematician his interests are functional analysis, specifically differentiability theory of convex and locally Lipschitz functions. As a chemical engineer he has worked on heat transfer and fluid flow problems. Therefore what he knows about combinatorial designs and graph factoring would fit on the back of an envelope. Other interests include ancient literature and Warhammer. He is a decidedly B-grade tennis player.