Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner number 33. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Gazette of the Australian Mathematical Society, School of Science, Information Technology & Engineering, University of Ballarat, PO Box 663, Ballarat, Vic. 3353, Australia.

The deadline for submission of solutions for Puzzle Corner 33 is 1 September 2013. The solutions to Puzzle Corner 33 will appear in Puzzle Corner 35 in the November 2013 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Same sum

Let S be a set of 10 distinct positive integers no more than 100. Prove that S contains two disjoint non-empty subsets which have the same sum.

Knights and knaves

In the following problems, *knights* always tell the truth and *knaves* always lie.

(i) There is a queue of people, each of whom is either a knight or a knave. It is known that there are more knights than knaves. Apart from the first person, every person points to someone in front of them in the queue and declares

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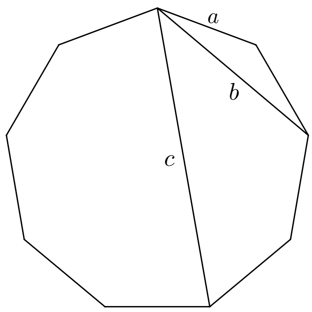
This puzzle corner is also featured on the Mathematics of Planet Earth Australia website
http://mathsofplanetearth.org.au/

the status of that person (being a knight or a knave). Is it possible for a bystander to determine the actual status of everyone in the queue?

(ii) There is a group of people, each of whom is either a knight or a knave. Each person makes the following two statements: 'All my acquaintances know each other', and 'Among my acquaintances, the number of knights is no more than the number of knaves.' We assume that knowing is mutual. Prove that the number knaves in the group is no more than the number of knights.

Diagonal difference

In a regular nonagon, prove that the length difference between the longest diagonal and the shortest diagonal is equal to the side length. In other words, prove $c-b=a$ in the diagram below.



Scissors and shapes 2

Edward is playing with scissors again. At each move, he chooses a polygon in front of him, and cuts it into two polygons with a single straight cut. Starting with a single rectangle, determine the minimal number of cuts required to obtain, among other shapes, at least 106 polygons with exactly 22 sides.

Photo: Marius Muresan, SXC

Perfect recovery

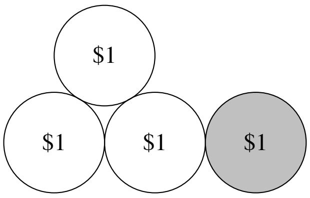
There are n distinct non-negative integers written on the board. Jack memorises these numbers before erasing them and replacing them with the $\binom{n}{2}$ pairwise sums. Now Jill enters the room and studies the sums on the board. Find all positive integers n for which it is possible for Jill to recover the original n integers uniquely.

Solutions to Puzzle Corner 31

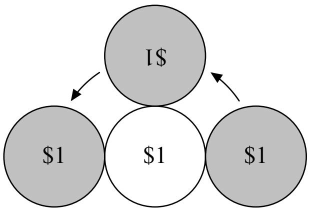
Many thanks to everyone who submitted. The \$50 book voucher for the best submission to Puzzle Corner 31 is awarded to Joe Kupka. Congratulations!

Rolling in riches

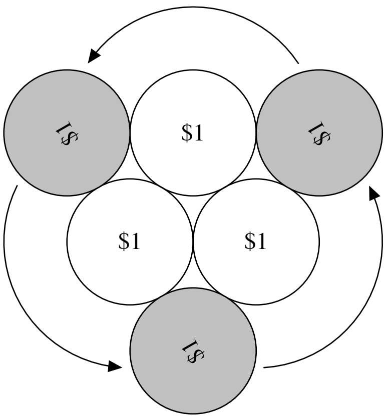
Place four \$1 coins as shown in the diagram below. Now roll the shaded coin anticlockwise around the other three, touching them the entire time, until it returns to the original position. How much has the shaded coin rotated relative to its centre?



Solution by M.V. Channakeshava: First note that if we rotate the shaded coin halfway around another coin while keeping firm contact, the shaded coin has actually rotated 360° about its centre. (Please try it out if you are not convinced!)



Now the required movement consists of three such half turns. Hence the shaded coin has rotated $3 \times 360° = 1080°$ relative to its centre.



Comment: Here is an interesting generalisation. Position n \$1 coins so that their centres form a convex n-gon whose side lengths are equal to the diameter of the coins. If a shaded \$1 coin is rolled one lap around the n coins while maintaining contact, prove that it has rotated $(n+6)\times 120^\circ$ relative to its centre.

Picky padlocks

An ancient scroll is kept in a chest, which is locked by a number of padlocks. All padlocks must be unlocked in order to open the chest. Copies of the keys to the padlocks are distributed amongst 12 knights, such that any group of 7 or more knights can open the chest should they choose to do so, but any group of less than 7 cannot. What is the minimal number of padlocks required to achieve this?

Solution by Pratik Poddar: Let us consider the general case where there are n knights in total, denoted by $T_1, T_2, \ldots, T_n$. We require every set of k knights to be able to open the chest, but no set of $k-1$ knights can open it.

Suppose a set of S of $k-1$ knights are trying to open the chest. There must be at least one lock, say L_S, which remains locked. Now if any additional knight not in S joins the group, he must be able to unlock L_S as the group of k knights must be able to open the chest. In other words, a knight can unlock L_S if and only if he doesn't belong to the set S.

By the same argument, for any set of $k-1$ knights, there exists a lock identified by the set. Hence we must have at least $\binom{n}{k-1}$ locks in total. To show that $\binom{n}{k-1}$ locks are sufficient, simply biject the locks to the sets of $k-1$ knights. In particular, give a key to a knight if and only if he is not in the set corresponding to the lock.

Therefore the required answer is $\binom{12}{7-1} = 924$.

Stargazing

An astronomer observed 20 stars with his telescope. When he added up all the pairwise distances between the stars, the result was X. Suddenly a cloud obscured 10 of the stars. Prove that the sum of the pairwise distances between the 10 remaining stars is less than $\frac{1}{2}X$.

Bonus: *Can you improve the bound? What is the smallest real number r such that the new sum is always less than rX, regardless of the configuration of the stars?*

Solution by Jensen Lai: We assume that all the stars are in distinct positions. Let the 10 obscured stars be $A_1, A_2, \ldots, A_{10}$ and the 10 unobscured stars be $B_1, B_2, \ldots, B_{10}$. By the triangle inequality

$$d(B_i, B_j) \le d(B_i, A_k) + d(A_k, B_j), \qquad 1 \le i, j, k \le 10, \tag{1}$$

where $d(\cdot,\cdot)$ is the distance function. Now let $k \equiv i + j \pmod{10}$ and sum (1) over all pairs of (i, j) satisfying $1 \le i < j \le 10$. By the restriction on k, each occurrence of $d(B_i, A_k)$ and $d(A_k, B_j)$ on the right-hand side of (1) is unique. Together, they capture 90 out of the 100 possible terms of the form $d(A_m, B_n)$. Hence we have

$$\sum_{1 \le i < j \le 10} d(B_i, B_j) \le \sum_{1 \le m, n \le 10} d(A_m, B_n).$$

Therefore

$$\begin{aligned} X &= \sum_{1\le i<j\le 10} d(B_i,B_j) + \sum_{1\le m,n\le 10} d(A_m,B_n) + \sum_{1\le i<j\le 10} d(A_i,A_j) \\ &> 2\sum_{1\le i<j\le 10} d(B_i,B_j), \end{aligned}$$

as required.

Bonus solution by Joe Kupka: We sum the inequality (1) over all triples (i,j,k) satisfying $1 \le i < j \le 10$ and $1 \le k \le 10$. In the summation, each term on the left-hand side appears 10 times while each term on the right-hand side appears 9 times. Hence

$$10 \sum_{1\le i<j\le 10} d(B_i,B_j) \le 9 \sum_{1\le m,n\le 10} d(A_m,B_n).$$

Therefore

$$\begin{aligned} X &= \sum_{1\le i<j\le 10} d(B_i,B_j) + \sum_{1\le m,n\le 10} d(A_m,B_n) + \sum_{1\le i<j\le 10} d(A_i,A_j) \\ &> \sum_{1\le i<j\le 10} d(B_i,B_j) + \frac{10}{9}\sum_{1\le i<j\le 10} d(B_i,B_j) \\ &= \frac{19}{9}\sum_{1\le i<j\le 10} d(B_i,B_j). \end{aligned}$$

So the improved lower bound is $\frac{19}{9}X$. To show that it is possible to get arbitrarily close to this bound, consider the limiting case of $A_1,\dots,A_{10},B_1,\dots,B_9$ being at the same point and B_{10} being a distance of d away. Then $X = 19d$ and the sum of the distances between $B_1,\dots,B_{10}$ is $9d$.

Golden creatures

At the beginning of time, in a galaxy far, far away, the Queen of Heaven gives birth to 40 golden creatures. On the last day of each year the King of Heaven sacrifices a randomly chosen creature to his own glory. After every 20 sacrifices, the Queen gives birth to 20 new creatures. Every creature lives until it is sacrificed by the King. Any creature who reaches 100 years of age receives a congratulatory letter from the Queen.

(i) *What is the probability that a creature will receive a congratulatory letter?*

(ii) *How many congratulatory letters, on average, will the Queen write in the first 1000 years?*

(iii) *One of the first 40 creatures is named Adam. One of the 20 creatures born after 40 sacrifices is named Eve. What is the probability that Adam will outlive Eve?*

Solution by Dave Johnson:

(i) In each 20 year cycle, exactly half of the 40 creatures present at the beginning survive the 20 years. So the chance of a particular creature surviving a

20 year cycle is $\frac{1}{2}$. In order to receive a congratulatory letter, a creature has to survive five of these cycles. So the probability is $\frac{1}{2^5} = \frac{1}{32}$.

(ii) In order for a creature to be eligible for a congratulatory letter in the first 1000 years, it must be born before year 901. In total, there are

$$40 + \underbrace{20 + 20 + \cdots + 20}_{44} = 920$$

such creatures. So the expected number of congratulatory letters is $\frac{920}{32} = 28.75$. Note that we have not included those creatures who turn 100 at the beginning of year 1001.

(iii) For Adam to outlive Eve, we require Adam to be alive when Eve is born. The chance of Adam surviving the first 40 years (two 20 year cycles) is $\frac{1}{2^2} = \frac{1}{4}$. From year 41 onwards, Adam and Eve have an equal chance of being sacrificed each year. So the probability of Adam outliving Eve is $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$.

Uncovered construction

Can you construct a set of 100 rectangles, with the property that not one of the rectangles can be completely covered by the the other 99?

Solution by Joe Kupka: We will construct an infinite sequence of rectangles $R_1, R_2, \ldots$, such that no one rectangle can be covered by the others. For $n = 1, 2, \ldots$, define the length l_n and width w_n of R_n by

$$l_n = 2^n, \quad w_n = \prod_{i=1}^{n} \frac{1}{2^{i+1} + 1}.$$

Note that the widths satisfy the following property

$$\begin{aligned} 2^{n+1} w_n &= 2^{n+1} \prod_{i=1}^{n} \frac{1}{2^{i+1}+1} \\ &= \left(1 - \frac{1}{2^{n+1}+1}\right) \prod_{i=1}^{n-1} \frac{1}{2^{i+1}+1} \\ &= \prod_{i=1}^{n-1} \frac{1}{2^{i+1}+1} - \prod_{i=1}^{n} \frac{1}{2^{i+1}+1} \\ &= \begin{cases} 1 - w_1, & n = 1, \\ w_{n-1} - w_n, & n > 1. \end{cases} \end{aligned}$$

Hence we have

$$\sum_{i=n}^{\infty} 2^{i+1} w_i = \begin{cases} 1, & n = 1, \\ w_{n-1}, & n > 1. \end{cases} \tag{2}$$

We now claim that our sequence of rectangles satisfy the required property. For the sake of contradiction, assume there exists a covering of R_i by the other rectangles. For each $j < i$, consider how much R_j is able to cover along the length of R_i. The longest length R_j can cover equals to the diagonal of R_j, which is strictly bounded above by $l_j + w_j$. In other words, the portion of R_i covered by R_j must be smaller than a $(l_j + w_j) \times w_i$ rectangle. Summing the areas covered by R_j over all $j < i$ and applying (2), we have

$$\sum_{j=1}^{i-1}(l_j + w_j)w_i < \left(\sum_{j=1}^{i-1} 2^j + \sum_{j=1}^{\infty} w_j\right) w_i < \left((2^i - 2) + 1\right) w_i = (2^i - 1)w_i.$$

Now for $k > i$, the areas covered by R_k are bounded above by $l_k w_k$. Summing these and again applying (2), we have

$$\sum_{k=i+1}^{\infty} l_k w_k = \sum_{k=i+1}^{\infty} 2^k w_k = \frac{1}{2} \sum_{k=i+1}^{\infty} 2^{k+1} w_k = \frac{1}{2} w_i.$$

Hence in total, the area of R_i covered by the other rectangles is bounded above by

$$\sum_{j=1}^{i-1}(l_j + w_j)w_i + \sum_{k=i+1}^{\infty} l_k w_k < (2^i - 1)w_i + \frac{1}{2} w_i < 2^i w_i = l_i w_i.$$

Since the area of R_i is $l_i w_i$, this is a contradiction. Therefore it is not possible to cover R_i by the other rectangles for every i, completing the solution.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.