Problems for the Mathematics Olympiads: Please donate generously

Hans Lausch*

About once in three years the Senior Problems Committee of the Australian Mathematical Olympiad Committee (AMOC) turns to our mathematical community with an appeal for problem donations that can be used in national, regional and international senior-secondary-school mathematics competitions. The latest appeal [2] provided examples of competition problems that had been set for various contests in Australia and in the Asia-Pacific region between 2008 and early 2010. The present article is to repeat this exercise with problems from competitions held between 2010 and early 2013. Problems chosen for these competitions are from 'pre-calculus' areas such as geometry (with a strong preference for 'Euclidean' geometry), number theory, algebra and combinatorics.

Before exhibiting various sample problems, it may be appropriate to put the various competitions serviced by the AMOC Senior Problems Committee into context.

1. The AMOC Senior Contest is held in August of each year. About 75 students, most of them in year 11, are given five problems and four hours to solve them.
2. The Australian Mathematical Olympiad (AMO) is a two-day event in February with about 100 participants, including 12 students from New Zealand. On either day, students face a four-hour paper containing four problems.
3. The Asian Pacific Mathematics Olympiad (APMO) takes place in March. The contest is a four-hour event with five problems to be solved. About 30 countries now take part in the APMO. Usually, 25 to 30 Australian students are invited to participate in this competition.
4. The International Mathematical Olympiad (IMO), initiated in 1959 with eight and now attended by students from some 100 countries, is the top secondary-school competition globally.

We now present sample problems from the above contests.

AMOC Senior Contest 2012, Problem 1 (difficulty level: low to medium).
Let $ABCD$ be a cyclic quadrilateral. Let K_1 be the circle that passes through D and is tangent to AB at A, and let K_2 be the circle that passes through D and is tangent to BC at C. Let P be the point other than D in which K_1 and K_2 intersect. Prove that P lies on the line through A and C.

*School of Mathematical Sciences, Monash University, VIC 3800.
Email: hans.lausch@monash.edu

AMOC Senior Contest 2011, Problem 5 (difficulty level: very high).
In a group of 2011 people of different heights, Zelda is the 27th tallest person. Let n be the number of ways that this group can form a queue so that Zelda is shorter than everyone ahead of her in the queue. Prove that there is exactly one set S of 2010 distinct positive integers such that n is the product of the elements of S.

The problem was submitted by Ian Wanless (Melbourne).

Australian Mathematical Olympiad 2012, Problem 5 (difficulty level: low).

(a) Prove that there is exactly one triple (x, y, z) of *real* numbers satisfying all three equations:

$$(x+20)(y+12)(z-4) = 1215$$
$$(x+20)(y-12)(z+4) = 1215$$
$$(x-20)(y+12)(z+4) = 1215.$$

(b) Determine the smallest integer n greater than 1215 such that the system

$$(x+20)(y+12)(z-4) = n$$
$$(x+20)(y-12)(z+4) = n$$
$$(x-20)(y+12)(z+4) = n.$$

has *integer* solutions x, y, z.

Australian Mathematical Olympiad 2013, Problem 2 (difficulty level: medium).
Determine all triples of positive integers that satisfy

$$20x^x = 13y^y + 7z^z.$$

Australian Mathematical Olympiad 2012, Problem 8 (difficulty level: high).
Two circles C_1 and C_2 intersect at distinct points A and B. Let P be the point on C_1 and let Q be the point on C_2 such that PQ is the common tangent closer to B than to A. Let BQ intersect C_1 again at R, and let BP intersect C_2 again at S. Let M be the midpoint of PR, and let N be the midpoint of QS. Prove that AB bisects $\angle MAN$.

The problem was submitted by Ivan Guo (Sydney).

Asian Pacific Mathematic Olympiad 2010, Problem 2 (difficulty level: medium).
For a positive integer k, call an integer a *pure kth power* if it can be represented as m^k for some integer m. Show that for every positive integer n there exist n distinct positive integers such that their sum is a pure 2009th power and their product is a pure 2010th power.

The problem was submitted by Angelo Di Pasquale (Melbourne).

International Mathematical Olympiad 2012, Problem 2 (difficulty level: medium). Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that $(1 + a_2)^2(1 + a_3)^3 \cdots (1 + a_n)^n > n^n$.

The problem was submitted by Angelo Di Pasquale (Melbourne).

The complete set of AMO problems and solutions covering the period 1979–2011 can be perused in [4] and [5], whereas the problems and solutions of all APMOs between 1989 and 2000 have appeared in [3]. Furthermore, the problems, including solutions and statistics, of each year's AMOC Senior Contest, the AMO, the APMO, the IMO and of other school mathematics competitions run by the AMOC are available in the AMOC's year books [1].

Please let me have your problem donation(s). As always, credit to the donor of successful problems will be given in [1].

References

[1] Brown, P.J., Di Pasquale, A. and McAvaney K.L. (appears annually). *Mathematics Contests: The Australian Scene*.

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[5] Lausch, H., Di Pasquale, A., Hunt, D.C. and Taylor, P.J. (2011). *Australian Mathematical Olympiads 1996–2011.* Australian Mathematics Trust, Canberra, 2011.