Book Reviews

Mathematical Excursions to the World's Great Buildings

Alexander J. Hahn
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This fascinating book recounts the history of monumental buildings in the Western tradition, interspersed with the mathematics pertinent to their design and construction.

The primary narrative focuses on the architectural form and structure of iconic buildings ranging from the Parthenon in Athens to the Sydney Opera House. The tone of the volume is set by the stunning cover, a photo of the Opera House at sunset with background a working drawing of elements of its design.

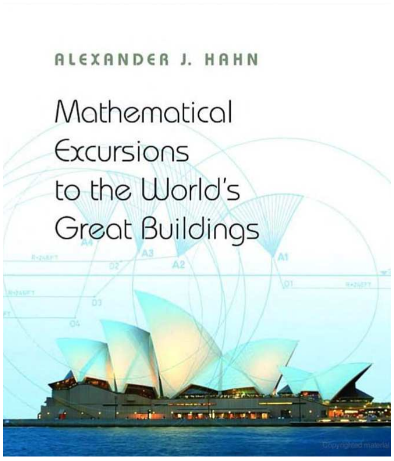
The secondary narrative concerns the mathematics behind the architecture, including plane and solid Euclidean geometry, analytic geometry, vector algebra and some basic calculus. These mathematical snippets are not systematically developed but scattered throughout the text in appropriate places determined by the architectural narrative. More advanced mathematics needed for the proper analysis of the structures, such as fluid mechanics, splines and finite element methods, are mentioned but not presented.

Greek architecture was based on aesthetics rather than structural analysis. The mathematical background is elementary geometry, including the Golden Mean and the construction of regular polygons, although the construction of the Acropolis buildings (−400) such as the Parthenon as well as the enormous conical theatre of Epidaurus (−360) preceded the formalisation of Euclid (−300). No doubt there was a deal of trial and error involved, as well as intuition and experience, but there remains no record of study of strength of materials or stress analysis.

The most important architectural advances of the Romans were undoubtedly the semi-circular arch and the dome, for example the Pantheon (120). Here an analysis of the stresses induced by the structure and the loads it carried would seem to be essential, and in fact the treatise of Vitruvius (−100) contains some of the theory needed. For example, Vitruvius describes the thickness of the piers necessary to counteract the horizontal component of the stress in an arch, as well as the use of high-density concrete in the lower parts of the structure and lower density in the higher. The success of Roman construction methods is attested by the massive aqueducts still standing in Italy and the colonies. Another innovation due to the Romans was the use of ovals rather than circles in stadiums and buildings, for example the Colosseum. The Roman oval was formed by four arcs of circles of two different radii rather than true ellipses, perhaps for ease of laying out. The relevant mathematics described here includes this geometric construction, as well as the decomposition of forces into components, and its inverse, the vector resultant of forces.

The construction of ever larger domes was a feature of ecclesiastical buildings in the Middle Ages, both Islamic and Christian. The book describes several notable examples, including the Hagia Sofia in Istanbul (535), a cathedral later converted to a mosque and the Great Mosque in Cordoba (800), a mosque later converted to a church. (The dates given in this review are only approximate, because construction often extended over hundreds of years, in many cases requiring extensive reconstructions due to damage by structural failure, fire or earthquakes.) The Hagia Sofia is one of the earliest examples of reinforced concrete construction, in the sense that circular chains were incorporated into the masonry during repairs, to balance the hoop stress which had cracked the dome. The mathematics in this section includes stress analysis in an arch and the symmetries involved in Islamic decoration.

The architectural innovations of the of the later Middle Ages are the splendid Gothic cathedrals characterised by ribbed vaults consisting of several crossing Gothic arches with a common vertex, flying buttresses and the large stained glass rose windows, a notable example being the Cathedral of Notre Dame in Chartres (1220). The mathematical section includes stress analysis in a pin-jointed truss, and the geometry of Gothic arches.



The flower of the Renaissance is the Cathedral of Florence, whose ribbed vault dome was designed by Brunelleschi (1377–1446). This 45 m-wide dome is supported by the structural strength of the Gothic arches themselves, without relying on buttresses. Another important example is the Duomo of Milan (1420), because we have extensive records of the Building Council which oversaw the design and construction. Others are St Mark's Basilica (1063) and the Doge's Palace (1309) in Venice, and the Cathedral and the Leaning Tower in Pisa (1300). The mathematical examples in this section include true ellipses, the geometry of the regular polytopes and decimal arithmetic. On the applied side, we have the study of strength of materials and stress in beams, arches and domes including an analysis of the stress distribution in the dome of the Florence Cathedral.

The High Renaissance saw Palladio's (1508–1580) villas and churches, and St Peter's Cathedral in Rome (1500), due to Bramante (1444–1515) and Michaelangelo (1475–1564), the oval Colonnade of St Peter's Square being designed by Bernini (1598–1670). The mathematical discussion in this section is devoted to perspective and conic sections.

Architecture in the Age of Reason is typified by St Paul's Cathedral in London. Its architect Wren (1632–1723) was familiar with the dictum of Hooke (1635–1703): 'As hangs the flexible line, so but inverted will stand the rigid arch'. Wren's design of St Paul's has a triple dome: the inner and outer dome are decorative hemispheres, but the load-bearing intermediate dome approximates a surface of revolution based on an inverted catenary. This method was later copied in the domes of St Isaac's Cathedral in St Petersburg, as well as the Capitol in Washington. Other innovations of this era include the use of cast and wrought iron in construction.

The design and construction of the Sydney Opera House occurred at a crucial time (1957–1973) in the history of architecture. Pre-stressed concrete was widely accepted; computers were enabling massive computations; software for finite element methods was developed. On the other hand, computer-aided design and manufacture (CAD–CAM) was not yet available, so Utzon's imaginative free-form roof design proved impossible to analyse mathematically. After experimenting with various geometries, Utzon, in conjunction with structural engineer Ove Arup, eventually settled on a design based on spherical triangles cut from a sphere of radius 75 m. As well as finite element analysis, scale models of the roof were tested in wind tunnels. The innovations in the design were not limited to the sail structure of the roof, but also extended to the podium. This was a concourse of area 116 m by 186 m largely unimpeded by columns. The roof was supported by pre-stressed concrete beams of length 75 m and width 2 m whose cross-section varied continuously from U-shaped at the supports, where shear stress is maximal, to T-shaped at the centre, where bending stress is maximal. The mathematics in this chapter is mainly concerned with spherical geometry but also includes the analysis of a hanging chain, both free-hanging and loaded, and the stress distribution in its inverted version, like Saariinen's Gateway Arch in St Louis (1965).

The book contains a final chapter wholly concerned with calculus and its application to building design and construction. Because of its lack of an underlying theory of limits, it should be regarded mainly as a review addressed to an audience who have already studied calculus and are interested in its applications.

To summarise, this intriguing book whose author, Alexander Hahn, is a professor of mathematics at the University of Notre Dame, Indiana, should be of interest to mathematicians who want to understand how their subject has been applied to produce some of the marvelous monuments of the world, and to amateurs of the history of architecture who wish to gain a deeper understanding of the mathematics behind some of those buildings.

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Archimedes' Modern Works

Bernard Beauzamy
Société de Calcul Mathématique SA, 2012, ISBN 978-2-9521458-7-9

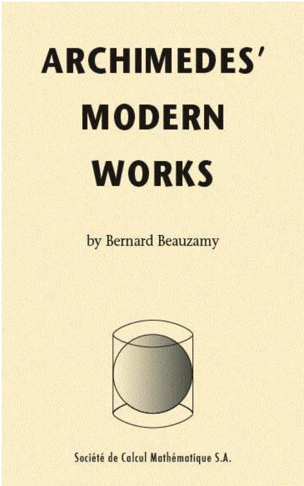
The quirky title of this book is an indication both of its content and its idiosyncratic style. Beauzamy is a French independent mathematician who is the founder and CEO of the mathematical consultancy company Société de Calcul SA, which also publishes his books, the topics of which range from the geometry of Banach spaces to applied probability.

The theme of this book is 'How to read Archimedes in the 21st Century'. The author analyses two results of Archimedes, the calculation of the surface area of a spherical cap, and the so-called 'Method' for comparing the volumes of a sphere, cone and cylinder. He then interprets these results in terms of recent mathematical developments.

The first result comes from Archimedes' work 'On the Sphere and the Cylinder, Book 1', showing that the area of the surface of a hemisphere is four times the area of the disc bounded by the equator of the sphere. Archimedes proves this by slicing the hemisphere by parallel vertical planes spaced so that they cut the equator in points which form the vertices of a regular polygon with an even number of sides. He approximates these slices by trapezoids which unfold to cover a regular polygon inscribed in a circle whose radius is twice that of the sphere. Archimedes uses the method of exhaustion to show firstly that by judiciously choosing the number of slices, the bent trapezoids approximate the surface of the hemisphere to any desired accuracy, and secondly that when unbent, they approximate the disc to any desired accuracy. In place of a modern notion of continuity, he uses trichotomy and a hidden lemma stating that between any two magnitudes $a < b$ of the same type, there is a third magnitude c such that $a < c < b$.

Beauzamy's insight is that the correspondence between points on the hemisphere and points on the disc is actually a measure preserving transformation; in fact it is identical to Lambert's 1772 Azimuthal Projection, used in cartography to map the terrestrial hemisphere onto a circle so that regions with the same spherical area have the same area on the map. Beauzamy calls a measure preserving map between measure spaces an 'Archimedes Map', and uses Archimedes Maps to solve various problems in Operations Research such as optimal allocation of resources and optimal placement of surveillance points. He also discusses the case of measure preserving but dimension decreasing transformations of manifolds.

The second result of Archimedes analysed by Beauzamy is the so-called 'Method'. This is not a mathematical theory, but rather a heuristic by which Archimedes found the relations between the volumes and surface areas of spheres, cones and cylinders, later proving them mathematically in 'On the Sphere and the Cylinder, Book 2'. The idea of the Method is to take for example a sphere and a cylinder of the same radius and height and of the same uniform density, and to calculate where to place the fulcrum so that they balance when placed at the ends of a see-saw. Beauzamy's take on this is to use the Method in various novel ways such as calculating centres of gravity of non-homogeneous bodies, solving systems of polynomial equations, calculating probabilities and non-destructive testing.



So far I have only mentioned positive aspects of this work. Unfortunately, there are also negative aspects that mar its appeal. The author's approach is avowedly ahistorical, which is acceptable when used for his declared aims, but not when he criticizes other Archimedes commentators. The standard modern edition of Archimedes is J.L. Heiberg's 'Archimedis opera omnia cum commentariis Eutocii' (1880), which is a transcription of two 16th Century codices (which do not contain the 'Method') and a 10th Century palimpsest (which does), together with a Latin translation. A few years later, T.L. Heath translated Heiberg's edition into English in 'The Works of Archimedes' (1912), adding his own commentary. Since then, several editions in various languages have appeared, the latest being 'The Archimedes Codex' (2004), by R. Netz and W. Noel, based on the recently rediscovered palimpsest.

Beauzamy's book denigrates the works of Heiberg and Heath, characterising the authors as a philologist and a school master with insufficient mathematical skills to understand Archimedes' works. However, it is hard to believe that a mathematician, were he or she not also a philologist, would have been able to edit and translate into a modern language the Doric codices of Archimedes' works. In particular, Heath, as well as Charles Mugler, editor of a 2003 French edition and translation of the two works, are accused of mathematical mistakes. The only evidence presented is the use of the word 'concave' in place of 'convex'. It is instructive to trace this supposed error. A literal translation of Archimedes, due to Dijksterhuis in 'Archimedes' (1956), is:

> I call such a line bent on the same side (ἐπὶ τὰ αὐτὰ κοίλη) if it has the property that when any two points on it are taken, the segment joining these points falls on the same side of the line, or touches it, but not on the other side.

Heath renders the parenthetic phrase as 'concave in the same direction' as do later translations such as those of Dijksterhuis and Netz. Since Archimedes is writing about curves and surfaces, not functions, this is a perfectly acceptable locution. A brief look at a few pages of Heath with their scholarly notes shows that his mathematical expertise is adequate to handle the mathematics of Archimedes and his commentator Eutocius. In a final ironic twist, Beauzamy praises Archimedes for the clarity of the diagrams in Heath's edition, apparently unaware that they do not occur in the codices and in fact are now attributed to Heiberg.

Another example of the dangers of an ahistorical approach to mathematical criticism is the book's charge of unethical behaviour if not plagiarism on the parts of Lambert, Newton and Leibniz. There is every reason to believe that Lambert's Azimuthal Projection is entirely original; indeed Beauzamy himself claims to be the first to recognise the connection to Archimedes Maps. While it is true that neither Newton nor Leibniz cite Archimedes in their works which introduce calculus, for reasons which I shall discuss below, they do acknowledge his priority in other works. Beauzamy's book claims that neither Leibniz, Newton nor Bourbaki see fit to cite Archimedes. This claim is ill-founded; the Index to Hoffmann's edition of Leibniz' 'Mathematische Schriften' lists over 60 references to Archimedes, of which three are to 'On the Sphere and the Cylinder'; Whitehead's index to the 'Principia Mathematicae' has over ten, including two to the latter work; and I found half a dozen Archimedes references in Bourbaki's 'General Topology' before stopping counting.

There are several reasons why neither Newton nor Leibniz cited Archimedes in their seminal papers 'A treatise on series and fluxions' (1711) and 'A new method for maxima and minima' (1684) respectively. Firstly, the areas and volumes calculated by Archimedes are those of conic sections, spirals and solids of revolution of conic sections, whereas both Leibniz and Newton deal with arbitrary smooth curves on a bounded interval, including exponential and trigonometric functions and their inverses. Secondly, Archimedes' techniques are equivalent to Riemann integration, whereas those employed by Leibniz and Newton are antidifferentiation and the Fundamental Theorem of Calculus. The full strength of Riemann integration is not required for smooth functions possessing elementary antiderivatives, which explains why it was not rediscovered until the 19th Century.

In the final chapter of this book, 30 pages are devoted to recounting well known apocryphal stories about the life and death of Archimedes. How much better his book would have been had Beauzamy instead elaborated on his perspicuous comments about Archimedes' deft manipulations of such notions as convexity, differentiability, centres of gravity, areas and volumes, even distinguishing between infinitesimal quantities of orders one and two. There could even have been a discussion of when and how Archimedes' discoveries were absorbed into later developments in mathematics.

In spite of my irritation at the polemical approach and disdain for historical research, I commend this book for its interesting and original insights into Archimedes' far-reaching discoveries.

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