Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner No. 29. Each Puzzle Corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes, SA 5095.

The deadline for submission of solutions for Puzzle Corner 29 is 21 December 2012. The solutions to Puzzle Corner 29 will appear in Puzzle Corner 31 in the March 2013 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Final product

Four real numbers can form six pairwise products. If five of the six products are 2, 3, 4, 5 and 6, what is the sixth product?

Bonus: Given that the four numbers are all positive, what are they?

Sticky stalemate

Adrian, Benny, and Christie are running for president in their club. On the ballot, each voter lists the three candidates in order of their preference. Counting only the first preferences results, dramatically, in a three-way tie. To break the deadlock, the second preferences are counted, but again there is a three-way tie.

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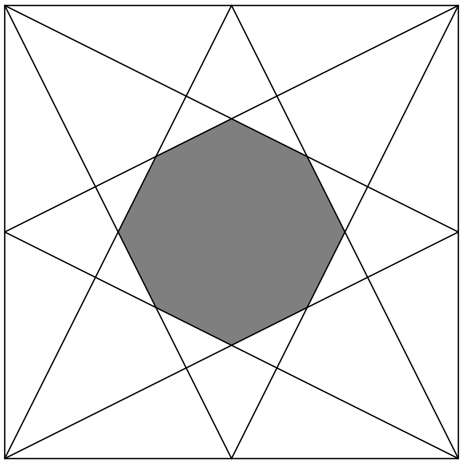
The astute Adrian notes that, since the number of voters is odd, they can make two-way decisions without ties. He proposes that the voters first choose between Benny and Christie, then the winner faces Adrian for the position.

The brainy Benny thinks it's a good resolution, since they only want to identify the winner, not the runner-up. The clever Christie disagrees and complains that this is giving Adrian an advantage. Who is right? Assuming the voters never change their preferences, what is Adrian's chance of winning under his proposed voting system?

Four-way intersection

Start with a unit square. Join each vertex to the midpoints of the two opposite sides. An octagon is formed as shown below.

What is the area of the octagon?



Three-point line

Every point on the real number line is coloured in one of two colours: red or blue.

1. Prove that there exist three real numbers $x < y < z$ such that they have the same colour and satisfy $z - y = y - x$.
2. Prove that for any positive real number r, there exist three real numbers $x < y < z$ such that they have the same colour and satisfy $z - y = r(y - x)$.

Revolving vault

A vault's door has a circular lock, which has n indistinguishable buttons on its circumference with equal spacing. Each button is linked to a light on the other side of the door, which is not visible from outside the vault. Each button toggles its linked light between on and off.

In the beginning, some number of lights are off and the door is locked. For each move, you are allowed to press several buttons simultaneously. If all lights are turned on as a result, the vault door will open. Otherwise, the circular lock will rotate to a random position, without changing the on/off status of each individual light. The rotation occurs quickly so it is impossible to track how much the lock has rotated.

Prove that it is always possible to open the door using a finite number of moves if and only if n is a power of 2.

Solutions to Puzzle Corner 27

Many thanks to everyone who submitted solutions. The $50 book voucher for the best submission to Puzzle Corner 27 is awarded to Dave Johnson. Congratulations!

Magic trick

The magician performs his famous trick: 'Think of a positive integer. Shuffle the digits to get a different number. Now calculate the difference between that and the original. Finally, delete the leading digit from the answer. Okay, tell me what you have and I will tell you what the deleted digit was'. How does he do it?


Photo: Ramzi Hashisho

Solution by Wu ChengYuan: The difference between the shuffled number and the original number will always be a multiple of 9.

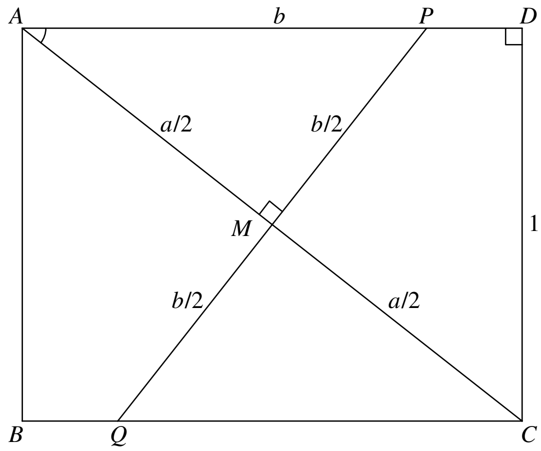
Using the fact that a number is divisible by 9 if and only if its digits sum up to a multiple of 9, the magician can sum up the digits of the answer, and take the difference between that and the next highest multiple of 9. The unique result will then reveal the deleted digit.

Crease length

A rectangular sheet of paper is folded once so that two diagonally opposite corners coincide. If the crease formed has the same length as the longer side of the rectangle, what is the ratio of the longer side to the shorter side?

Solution by Dave Johnson: Let the vertices of the rectangle be A, B, C, D and let the shorter side CD to be one unit long. Also let the length of AC be a and the length of AD be b. Hence the required ratio is simply b.

Since C coincides with A after the fold, they must be reflections of each other with respect to the crease PQ. Hence PQ is the perpendicular bisector of AC, going through M, the centre of the rectangle. By symmetry, $AM = a/2$ and $PM = PQ/2 = AD/2 = b/2$.



Triangles AMP and ADC are similar because they are both right-angled triangles sharing an acute angle at A. So

$$\frac{a/2}{b/2} = \frac{b}{1} \quad \Longrightarrow \quad a = b^2.$$

Applying Pythagoras' Theorem to triangle ADC, we get

$$a^2 = b^2 + 1^2 \quad \Longrightarrow \quad b^4 - b^2 - 1 = 0.$$

Solving the quartic yields

$$b^2 = \frac{1+\sqrt{5}}{2} \quad \Longrightarrow \quad b = \sqrt{\frac{1+\sqrt{5}}{2}}.$$

Therefore the required ratio is the square root of the golden ratio.

Breaking point

You have two identical crystal globes, either of which would break if dropped from the top of a 100-storey building. Your task is to determine the highest floor from which the globes can be dropped without breaking. What is the minimum number of drops required to do this? You may break both globes in the process.

Bonus: *What if you had three crystal globes?*

Solution by Joe Kupka: Consider the general situation with g globes and n drops. Let $h_g(n)$ be the largest positive integer such that it is possible to determine the breaking point (highest point the globes can be dropped without breaking) of an $h_g(n)$ floor building. Then $h_g(1) = 1$ since one drop can only determine one floor. Also $h_1(n) = n$, since with one globe, the only possible strategy is to try the floors one by one from the bottom to the top. We proceed by induction.

When $g, n \geq 2$, the first drop cannot be higher than floor $h_{g-1}(n-1) + 1$. This is because if the globe breaks, there are only $g - 1$ globes and $n - 1$ drops left, the highest floor we can still deal with is $h_{g-1}(n-1)$. If the first drop does not break, we can then treat floor $h_{g-1}(n-1)+2$ as the new floor 1, and reduce it to a problem with g globes and $n-1$ drops. Iterating this argument:

$$\begin{aligned} h_g(n) &= 1 + h_{g-1}(n-1) + h_g(n-1) \\ &= 2 + h_{g-1}(n-1) + h_{g-1}(n-2) + h_g(n-2) \\ &= 3 + h_{g-1}(n-1) + h_{g-1}(n-2) + h_{g-1}(n-3) + h_g(n-3) \\ &\quad\vdots \\ &= (n-1) + \sum_{i=1}^{n-1} h_{g-1}(i) + h_g(1) \\ h_g(n) &= n + \sum_{i=1}^{n-1} h_{g-1}(i). \end{aligned} \tag{1}$$

Note that the argument also inductively provides an algorithm to achieve $h_g(n)$.

When $g = 2$, (1) becomes

$$h_2(n) = n + \sum_{i=1}^{n-1} h_1(i) = n + \sum_{i=1}^{n-1} i = \frac{n(n+1)}{2}.$$

Since $h_2(13) = 91 < 99 \leq 105 = h_2(14)$, the required number of drops for two globes is 14.

Bonus: When $g = 3$, (1) becomes

$$h_3(n) = n + \sum_{i=1}^{n-1} h_2(i) = n + \sum_{i=1}^{n-1} \frac{i(i+1)}{2} = \frac{n(n^2+5)}{6}.$$

The last equality follows from the identity $\sum_{i=2}^{n} \binom{i}{2} = \binom{n+1}{3}$. Since $h_3(8) = 92 < 99 \leq 129 = h_3(9)$, the required number of drops for three globes is 9.

Better bets

Betty plays the following game: The cards from a shuffled deck are revealed to her one by one. Just before each card is shown, Betty can bet any portion of her wealth, with even odds, on the colour of the upcoming card. For example if Betty bets $10, then she could either win an additional $10 for guessing correctly, or lose the $10 for guessing incorrectly. This is repeated until the entire deck runs out. Starting with a single dollar, what's the greatest amount of wealth Betty can guarantee, by the end of the game?

Solution by Adrian Nelson: The greatest amount of wealth Betty can be guarantee to win starting with one dollar is

$$\frac{2^{52}}{\binom{52}{26}} \approx \$9.08\,.$$

Suppose Betty plays the game with a deck of r red cards and b black cards. Let $M(r, b)$ be the greatest amount she can guarantee to win starting with one dollar, then

$$M(r,b) = \frac{2^{r+b}}{\binom{r+b}{r}} = \frac{2^{r+b}}{\binom{r+b}{b}}. \tag{2}$$

We will prove this by induction on the number of cards $r+b$. If the deck is monochromatic, then Betty can double her money every step and achieve

$$M(r,0) = 2^r, \qquad M(0,b) = 2^b.$$

This establishes the base case.

Assume (2) is true for a deck of $r+b=n$ cards. We show the result for a deck of $r+b=n+1$ cards and $r,b \geq 1$. Suppose that Betty starts with one dollar and bets ρ dollars on the first card being red. Since, by symmetry, betting on black can be seen as a negative bet on red, we allow ρ to be negative and $-1 \leq \rho \leq 1$. Now for the first card:

- If a red card is drawn, the greatest amount she is guaranteed to win is
$$f(\rho) = (1+\rho)M(r-1,b).$$
- If a black card is drawn, the greatest amount she is guaranteed to win is
$$g(\rho) = (1-\rho)M(r,b-1).$$

Hence the greatest amount she is guaranteed to go on to win by betting on red is

$$m(\rho) = \min(f(\rho), g(\rho)).$$

We want to maximise $m(\rho)$ by adjusting ρ. Since f is increasing and g is decreasing, the maximum occurs when $f(\rho) = g(\rho)$, or

$$\begin{aligned} &(1+\rho)M(r-1,b) = (1-\rho)M(r,b-1) \\ \iff \quad &\rho = \frac{M(r,b-1) - M(r-1,b)}{M(r,b-1) + M(r-1,b)}. \end{aligned} \tag{3}$$

Note that in (3), ρ does indeed lie between -1 and 1. Substituting back, the maximum is

$$f\left(\frac{M(r,b-1) - M(r-1,b)}{M(r,b-1) + M(r-1,b)}\right) = \frac{2M(r,b-1)M(r-1,b)}{M(r,b-1) + M(r-1,b)},$$

or the harmonic mean of $M(r-1,b)$ and $M(r,b-1)$.

Using the inductive hypothesis with (2), and a standard binomial identity, we have

$$\begin{aligned} \frac{1}{M(r,b)} &= \frac{1}{2}\left(\frac{1}{M(r,b-1)} + \frac{1}{M(r-1,b)}\right) \\ &= \frac{1}{2}\left(\frac{\binom{r+b-1}{r}}{2^{r+b-1}} + \frac{\binom{r+b-1}{r-1}}{2^{r+b-1}}\right) \\ &= \frac{\binom{r+b}{r}}{2^{r+b}}, \end{aligned}$$

completing the induction.

Hidden catch

1. *There is a row of 10 rooms and a spy is in one of them. Each night, he moves to an adjacent room. You, the esteemed investigator trying to catch the spy, can only check one room per day. How do you catch him?*
2. *Now the row has infinitely many rooms, extending indefinitely in both directions. Each night, the spy moves n rooms to the right, where n is a fixed but unknown integer. (If n is negative then he actually moves to the left.) You, being as resourceful as ever, can still only check one room per day. Can you still catch the spy?*

Solution by David Angell:

1. On day k, let the spy's room number be S_k and the investigator's room number be I_k. Adopt the following strategy:
   - For days 1-8, let $I_1 = 2, I_2 = 3, \dots, I_8 = 9$;
   - For days 9-16, let $I_9 = 9, I_{10} = 8, \dots, I_{16} = 2$.

   If S_1 is even, then we will show $S_k = I_k$ for some $1 \leq k \leq 8$. First note that for $1 \leq k \leq 8$, S_k and I_k must have the same parity. Hence $S_k - I_k$ must be even and can change by at most 2 each day.
   Again by parity, we have $S_1 \neq 1, S_8 \neq 10$, which implies
   $$S_1 - I_1 \geq 0, \quad S_8 - I_8 \leq 0.$$
   So there must exist some k between 1 and 8 such that $S_k - I_k = 0$. Therefore the spy is caught within the first 8 days. If S_1 is odd, then by similar arguments, the spy must be caught between days 9 and 16. Note that this strategy generalises to $n \geq 2$ rooms, where the spy will be caught in $2n - 4$ days.
2. Let the spy's starting room number be S. Since $(S, n) \in \mathbb{Z} \times \mathbb{Z}$, there are only countably many sequences of spy movements. Hence there exists a one-to-one correspondence $b \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ from the possible pairs of (S, n) to the positive integers. Simply check room $S + (b(S, n) - 1)n$ on day $b(S, n)$ and the spy will be caught eventually.
   Note that this strategy generalises to any situations where the spy only has countably many possible paths.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.