Lift-off fellowship report: The smallest degree for a strict inequality of a direct product of finite groups

Neil Saunders*

Let G be a finite group. The minimal faithful permutation degree of G, denoted $\mu(G)$, is the smallest nonnegative integer n such that G embeds inside the symmetric group $\mathrm{Sym}(n)$. In [1], we gave a general exposition on this topic and stated a fundamental question of interest, namely:

What is the smallest degree for which one can find two groups G and H such that

$$\mu(G \times H) < \mu(G) + \mu(H)? \tag{1}$$

In [1], we exhibited a class of examples of groups that satisfy (1) which were all contained in the class of monomial reflection groups $G(p, p, q)$ for p and q distinct primes (see also [2] for more details). Specifically, G could be taken to be the group $G(p, p, q)$ and H to be the centraliser of G in its minimally embedded image inside $\mathrm{Sym}(pq)$. In this class, the smallest degree of a direct product of two groups satisfying (1) was 10 and was obtained by taking G to be the group $G(2, 2, 5)$ and thus H to be C_2.

We can now report that 10 is indeed the smallest degree where one can find groups satisfying (1) and that in $\mathrm{Sym}(10)$, there are five nonisomorphic examples. Moreover in each example, the group G contained a subgroup isomorphic to the split extension, $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5$ (the *deleted permutation module* for $\mathrm{Sym}(5)$ over $\mathbb{F}_2$ extended by a 5-cycle) and that $10 = \mu(G) = \mu(G \times H)$, where was H is isomorphic to C_2.

A central tool in the proof of this result was the class of finite groups $\mathcal{C}$ defined by Wright in [3]. Specifically, $\mathcal{C}$ is the class of finite groups with the defining property that for every G in $\mathcal{C}$, G contains a *nilpotent* subgroup G_1 such that $\mu(G) = \mu(G_1)$. It is a result of Wright's that

$$\mu(H \times K) = \mu(H) + \mu(K) \tag{2}$$

whenever the groups H and K are nilpotent, so it quickly follows that $\mathcal{C}$ is closed under taking direct products and that any two groups in $\mathcal{C}$ satisfy (2). Thus in proving that 10 was minimal in the sense of (1), it sufficed in many cases to assume

*School of Mathematics and Statistics, University of Sydney, NSW 2006.
Email: neil.saunders@sydney.edu.au

that G was not contained in $\mathcal{C}$ and in doing so, we discovered that this class is surprisingly pervasive for groups of minimal degree at most 9, as the following theorem states.

Theorem 1. Let $\mathcal{C}$ be as above.

- If $\mu(G) \leq 6$, then G is contained in $\mathcal{C}$.
- There are only 9 nonisomorphic groups of minimal degree at most 9 that are not contained in $\mathcal{C}$.

Of the six groups of minimal degree 9 that are not contained in $\mathcal{C}$, three of them are of *affine type* which lie in a more general family of *primitive permutation groups*. This gives a method to construct more groups that are not contained in $\mathcal{C}$, however systematically describing $\mathcal{C}$ will almost certainly turn out to be a difficult problem.

Thanks to a recent conversation (P. Cameron, July 2011, pers. comm.), the examples which satisfy (1) form building blocks in constructing groups which satisfy the following cascade of inequalities

$$\max\{\mu(G), \mu(H)\} < \mu(G \times H) < \mu(G) + \mu(H). \tag{3}$$

For example, let G be the group $G(2,2,5)$ as before and H be the group $C_2 \times C_2$. Then $\mu(G) = 10$ (by [2]), $\mu(H) = 4$ and we observe (by standard group theory arguments) that

$$G \times H \cong G(2,2,5) \times (C_2 \times C_2) \cong (G(2,2,5) \times C_2) \times C_2 \cong C_2 \wr \mathrm{Sym}(5) \times C_2.$$

Since both $C_2 \wr \mathrm{Sym}(5)$ and C_2 can be readily seen to be contained in $\mathcal{C}$, and (with some more standard group theory arugments) that $\mu(C_2 \wr \mathrm{Sym}(5)) = 10$, we have $\mu(G \times H) = 12$. Thus we have an example of groups satisfying (3). In fact, all examples of groups in the $G(p, p, q)$ family that satisfy (1) can be manipulated to construct examples that satisfy (3). Now a deeper and more sophisticated version of the original question posed at the beginning of this report, and previously in [1], can be asked:

Do there exist finite groups G and H that themselves do not decompose as a nontrivial direct product and satisfy (3)?

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References

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Neil completed both his undergraduate and postgraduate degrees at The University of Sydney. His PhD thesis was titled 'Minimal faithful permutation representations of finite groups' and was completed under the supervision of Associate Professor David Easdown and Associate Professor Anthony Henderson. After graduating in February 2011, he continued to work at The University of Sydney as a casual tutor, and later, as an Associate Lecturer. In September 2012, Neil took up a Heilbronn Research Fellowship at the University of Bristol.