Puzzle Corner

Ivan Guo*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner No. 30. Each Puzzle Corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes, SA 5095.

The deadline for submission of solutions for Puzzle Corner 30 is 28 February 2013. The solutions to Puzzle Corner 30 will appear in Puzzle Corner 32 in the May 2013 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Peculiar pace

Jodie jogged for 25 minutes. In any 10-minute period, her average speed was 18 kilometres per hour. How far did she run?

Rolling roadblocks

There are 10 cars on an infinitely long, single-lane, one-way road, all travelling at different speeds. When any car catches up to a slower car, it slows down and stays just behind the slower car without overtaking. Eventually, the cars form a number of separate blocks. On average, how many blocks do you expect to see?

*School of Mathematics and Statistics, University of Sydney, NSW 2006.
E-mail: ivanguo1986@gmail.com

Polynomial parity

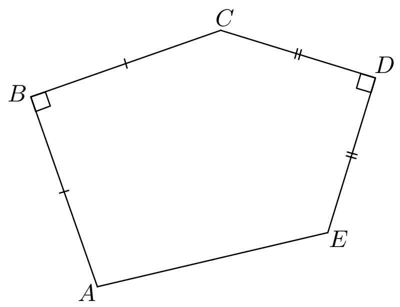
Submitted by Alexander Hanysz

(i) Let P and Q be complex polynomials with no common factors. Suppose the rational function P/Q is an even function. Prove that P and Q are both even functions.

(ii) Let P, Q and R be complex polynomials. Suppose PQ, PR and QR are all even functions. Prove that either P, Q and R are all even functions, or they are all odd functions.

Squaring a pentagon

Submitted by Rory Tarnow-Mordi



Let $ABCDE$ be a pentagon with $AB = BC$, $CD = DE$ and $\angle B = \angle D = 90°$. Can you cut the pentagon into three pieces and then rearrange them to form a square?

Team tactics

Submitted by Ross Atkins

In a game show, a team of n girls is standing in a circle. When the game starts, either a blue hat or a red hat is placed on the head of each girl. Due to the set-up of the stage, each girl can only see the hats of the two adjacent girls, but not her own hat nor the hat of anyone else. Without any communication, the girls have to simultaneously guess the colour of their own hats. The team wins if and only if everyone guesses correctly.

Before the show, the girls try to devise a strategy to maximise their probability of winning. What is the maximum probability of winning if

(i) $n = 3$?

(ii) $n = 4$?

(iii) $n = 5$?

Solutions to Puzzle Corner 28

Many thanks to everyone who submitted solutions. The \$50 book voucher for the best submission to Puzzle Corner 28 is awarded to Jensen Lai. Congratulations!

Mean marks

The final exam marks have just been released. Within each class, the boys have a higher average score than the girls. Does that necessarily mean the boys have a higher average score across the entire grade?

Solution by Shaun De Roza: This does not necessarily mean the boys have a higher average score across the entire grade. Consider the following set of marks from two classes:

Class 1: boy 0, boy 1, girl 0; boy average 0.5, girl average 0;
Class 2: boy 98, girl 96, girl 96; boy average 98, girl average 96.

In both classes the boy average is higher than the girl average. However, over the entire grade the boy average is 33, lower than the girl average which is 64.

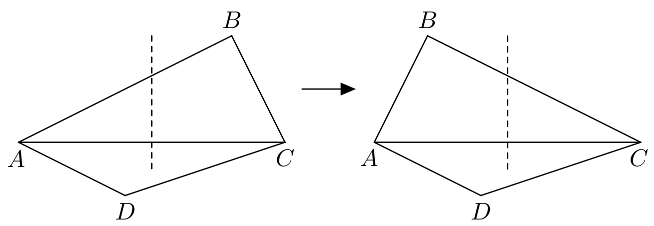
Enlarged enclosure

You have four straight pieces of fencing, that are 1, 4, 7 and 8 metres in length. What is the greatest area you can enclose with these pieces?

Solution by Jensen Lai: Let the four pieces of fence form quadrilateral $ABCD$.

If the quadrilateral is concave, say $\angle ABC$ is a reflex angle. Reflecting AB and BC across line AC yields a quadrilateral with the same side lengths but greater area. Therefore, the arrangement which encloses the greatest area is not concave.

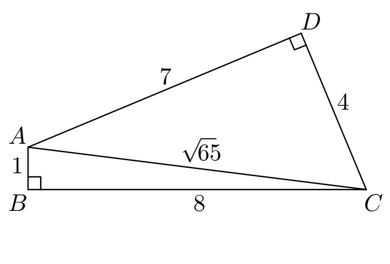
Given quadrilateral $ABCD$, if triangle ABC is reflected about the perpendicular bisector of AC (as shown below), the total area of the quadrilateral does not change. The same side lengths are utilised however pairs of sides which were previously opposite each other are now adjacent. Therefore, for every combination of fence order it is possible to obtain the greatest area.



Consider the quadrilateral $ABCD$ where

$$AB = 1, \qquad BC = 8, \qquad CD = 4, \qquad DA = 7.$$

The total area of the quadrilateral is the sum of triangles ABC and ACD. Using the formula $Area = \frac{1}{2}ac \sin B$, the area of ABC is maximised when B is a right angle. Similarly, the area of ACD is maximised when D is a right angle. It is possible to achieve both limits simultaneously since $1^2 + 8^2 = 4^2 + 7^2 = 65$ so both right-angled triangles would have a hypotenuse of $\sqrt{65}$. Therefore the maximum area which can be enclosed by the four pieces is $(1 \times 8 + 4 \times 7)/2 = 18\text{m}^2$.



Note: The question does not state that the pieces must be placed end to end. However, even if you could attach the end of one fence part way along another, the above analysis still holds.

Comment by Dave Johnson: By a simple trigonometric calculation, it can be shown that the area of a quadrilateral with four fixed side lengths is maximised when the quadrilateral is cyclic.

Flipping fun

There is a coin at every integer point of the number line. A stencil with a finite set of fixed holes at integer distances is chosen. The stencil may be moved along the number line, and for any fixed position of the stencil, one may simultaneously flip all the coins accessible through the holes. Initially all coins are showing heads. Prove that for any stencil it is possible to get exactly two tails after a finite number of such operations.

Solution by Natalie Aisbett: The problem can clearly be solved when there are less than three holes in the stencil.

Assume that the stencil has $n \geq 3$ holes. Label the coins, from left to right, by the integers. Starting with every coin showing heads, we carry out the following algorithm:

- The first flip is taken with the leftmost hole over coin 1.
- Before the ith flip for $i > 1$, let the leftmost coin showing tails, apart from coin 1, be A_i. The ith flip is taken with the leftmost hole over coin A_i.
- After the ith flip, coin 1 is showing tails while all other tails are located in the interval $[A_i + 1, A_i + n - 1]$. Now proceed with the $(i+1)$th flip where $A_{i+1} > A_i$ and so on.

Let the sequence of heads and tails in the interval $[A_i + 1, A_i + n - 1]$ after the ith flip be S_i. Since for all i the length of S_i is fixed at $n-1$, there is only a finite number of possible patterns for S_i. Hence $S_1, S_2, \ldots$ must eventually repeat itself. Since the flipping algorithm is reversible, S_i uniquely determines both S_{i+1} and S_{i-1}. Therefore $S_1, S_2, \ldots$ is a periodic sequence.

Now the sequence S_1 shows tails at locations corresponding to the holes of the stencil, ignoring the leftmost hole. Take any $j > 1$ with $S_1 = S_j$. Right before the jth flip, the only coins showing tails must be coin 1 and coin $A_j > 1$, completing the solution.

Mad hat party

The Mad Hatter is holding a hat party, where every guest must bring his or her own hat. At the party, whenever two guests greet each other, they have to swap their hats. In order to save time, each pair of guests is only allowed to greet each other at most once.

After a plethora of greetings, the Mad Hatter notices that it is no longer possible to return all hats to their respective owners through more greetings. To sensibly resolve this maddening conundrum, he decides to bring in even more hat-wearing *guests, to allow for even more greetings and hat swappings. How many extra guests are needed to return all hats (including the extra ones) to their rightful owners?*

Solution by Ross Atkins: Only two additional guests are required, if the following method is used. Let these additional guests be helper A and helper B.

Step 1: Arrange the original partygoers into lines as follows. In each line, every person is wearing the hat of the person directly behind, except for the last person who is wearing the hat of the person at the front of that line.

Step 2: For a single line, first get helper A to swap hats with the person at the back of the line. Then get helper B to swap hats with each member of the line one at a time from front to back. Finally get helper A to swap hats with the person at the front of the line. The net result is the correction of the everyone in this line and an effective hat swap of helpers A and B.

Step 3: Repeat Step 2 for each line. Now all the original partygoers have their original hats, whilst helpers A and B have either their correct hats or each other's hats, depending on the parity of the number of cycles in the original permutation.

Step 4: Swap the hats of helpers A and B if necessary.

Chord variations

There are 2012 points on the circumference of a circle, dividing it into 2012 equal arcs. The points are to be labelled with $A_1, \ldots, A_{2012}$ in some order.

(i) *Can you label the points in a way so that no two of the 2012 chords*

$$A_1A_2\ ,\ A_2A_3\ ,\ \ldots\ ,\ A_{2011}A_{2012}\ ,\ A_{2012}A_1$$

are parallel?

(ii) *Can you label the points in a way so that no two of the 1006 chords*

$$A_1A_2\ ,\ A_3A_4\ ,\ \ldots\ ,\ A_{2009}A_{2010}\ ,\ A_{2011}A_{2012}$$

have equal length?

Solution by Jensen Lai:

(i) Number the points from 1 to 2012 in order around the circle. Every chord of the circle is assigned a value equal to the sum of numbers at its endpoints. We notice that two chords have the same value mod 2012 if and only if they are parallel. It is clear that there are exactly 2012 possible directions the chords can travel, namely the directions of the 1006 chords from point x to $x + 1$ for $1 \leq x \leq 1006$ and the 1006 diameters.
If the chords

$$A_1A_2\ ,\ A_2A_3\ ,\ \ldots\ ,\ A_{2011}A_{2012}\ ,\ A_{2012}A_1$$

contain 2012 uniquely orientated chords, then the sum of the chord values must be congruent to

$$1 + 2 + \cdots + 2012 = \frac{2012 \times 2013}{2} = 1006 \times 2013 \equiv 1006 \pmod{2012}.$$

However, each vertex appears exactly twice, so each vertex value is counted twice when summing the chord values. Hence the total value of the 2012 chords is

$$2 \times (1 + 2 + \cdots + 2012) = 2012 * 2013 \equiv 0 \mod 2012.$$

This is a contradiction. Therefore, it is not possible to label the points in a way so that no two of the 2012 chords are parallel.

(ii) There are 1006 possible lengths of chords defined by the number of small arcs between endpoints. They range from 1 arc to 1006 arcs (or a diameter of the circle). Therefore, if the chords

$$A_1A_2\ ,\ A_3A_4\ ,\ \ldots\ ,\ A_{2009}A_{2010}\ ,\ A_{2011}A_{2012}$$

contain 1006 different lengths, they must each span a different number of small arcs from 1 to 1006.

Colour all of the points around the circle alternating black and white yielding 1006 black points and 1006 white points, with no two adjacent points being the same colour. A chord of 'odd length' (spanning an odd number of small arcs) will connect one black point to one white point. Therefore, drawing 503 chords of 'odd length' leaves 503 black points unconnected. Meanwhile, a chord of 'even length' (spanning an even number of small arcs) will connect two black points or no black points, maintaining the parity of unconnected black points. Since 503 is an odd number, it is impossible to reduce the number of unconnected black points down to zero using only chords of 'even length'.

Therefore, it is not possible to label the 2012 points in such a way that no two of the 1006 resulting chords have the same length.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.