Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner No. 27. Each Puzzle Corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes, SA 5095.

The deadline for submission of solutions for Puzzle Corner 27 is 15 July 2012. The solutions to Puzzle Corner 27 will appear in Puzzle Corner 29 in the September 2012 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Magic trick

The magician performs his famous trick: 'Think of a positive integer. Shuffle the digits to get a different number. Now calculate the difference between that and the original. Finally, delete the leading digit from the answer. Okay, tell me what you have and I will tell you what the deleted digit was'. How does he do it?

Crease length

A rectangular sheet of paper is folded once so that two diagonally opposite corners coincide. If the crease formed has the same length as the longer side of the rectangle, what is the ratio of the longer side to the shorter side?

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Breaking point

You have two identical crystal globes, either of which would break if dropped from the top of a 100-storey building. Your task is to determine the highest floor from which the globes can be dropped without breaking. What is the minimum number of drops required to do this? You may break both globes in the process.

Bonus: What if you had three crystal globes?

Better bets

Betty plays the following game: The cards from a shuffled deck are revealed to her one by one. Just before each card is shown, Betty can bet any portion of her wealth, with even odds, on the colour of the upcoming card. For example if Betty bets \$10, then she could either win an additional \$10 for guessing correctly, or lose the \$10 for guessing incorrectly. This is repeated until the entire deck runs out. Starting with a single dollar, what's the greatest amount of wealth Betty can guarantee, by the end of the game?

Hidden catch

Submitted by Andrew Crisp

1. There is a row of 10 rooms and a spy is in one of them. Each night, he moves to an adjacent room. You, the esteemed investigator trying to catch the spy, can only check one room per day. How do you catch him?
2. Now the row has infinitely many rooms, extending indefinitely in both directions. Each night, the spy moves n rooms to the right, where n is a fixed but unknown integer. (If n is negative then he actually moves to the left.) You, being as resourceful as ever, can still only check one room per day. Can you still catch the spy?

Solutions to Puzzle Corner 25

Many thanks to everyone who submitted solutions. The \$50 book voucher for the best submission to Puzzle Corner 25 is awarded to Andrew Elvey Price. Congratulations!

Plugs and sockets

Assume there are n types of power plugs and n types of power sockets in the world. Each plug type is compatible with exactly one socket type, and vice versa. An adaptor converts one socket type to another. What is the minimum number of adaptors you would need in order to connect any plug type to any socket type? Keep in mind that it is possible to use multiple adaptors at the same time.

Solution by Alan Jones: Since we have to cope with n different plugs and n different sockets, we need at least n adaptors. Denote the types by 1 to n, and an adaptor by $[i, j]$ if it joins an i-socket to a j-plug. The set of n adaptors

$$\{[1,2],[2,3],\ldots,[n-1,n],[n,1]\}$$

allows the construction of an effective $[i, j]$ adaptor by connecting these sequentially (and cyclically if necessary) from $[i, i+1]$ to $[j-1, j]$. Therefore n is the minimum number required.

Factor of growth

Given a natural number X, you are allowed to choose one of its divisors $d > 1$ and add it to X. Then the operation is repeated on the sum $X + d$ and so on. If you start with $X = 4$, which natural numbers can you reach by these operations?

Solution by Ben Odgers: From $X = 4$ we can get to all even numbers greater than 4 by repeatedly adding 2. This includes all numbers of the form $2A$ where $A \geq 2$. Furthermore from $2A$ we can get to all numbers of the form BA for $B \geq 2$, by repeatedly adding the divisor A. Hence we can get to all numbers of the form AB for $A \geq 2, B \geq 2$, or all composite numbers.

That just leaves the prime numbers and 1, which we cannot reach because the result of the operation, $X + d$, always has a divisor of d between 1 and $X + d$.

Concocting convergence

Can you find a pair of sequences of positive real numbers

$$a_1 \geq a_2 \geq a_3 \geq \cdots, \quad b_1 \geq b_2 \geq b_3 \geq \cdots$$

such that the infinite sums $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ are unbounded, but the infinite sum $\sum_{i=1}^{\infty} \min\{a_i, b_i\}$ is bounded?

Bonus: *What if you are given $b_i = 1/i$?*

Solution by Joe Kupka: Yes, these sequences do exist. Divide the sequences up into intervals using indices $1 = k_1 < k_2 < \cdots$ and make them constant within each interval:

$$\begin{aligned} a_{k_1} &= a_{k_1+1} = \cdots = a_{k_2-1}, \qquad a_{k_2} = a_{k_2+1} = \cdots = a_{k_3-1}, \ \ldots \\ b_{k_1} &= b_{k_1+1} = \cdots = b_{k_2-1}, \qquad b_{k_2} = b_{k_2+1} = \cdots = b_{k_3-1}, \ \ldots \end{aligned}$$

Let the sums on each interval be

$$\begin{aligned} S_i &= a_{k_i} + \cdots + a_{k_{i+1}-1} = (k_{i+1} - k_i)a_{k_i}, \\ T_i &= b_{k_i} + \cdots + b_{k_{i+1}-1} = (k_{i+1} - k_i)b_{k_i}. \end{aligned} \tag{1}$$

It then suffices to achieve

$$\begin{array}{llllll} S_1 = 1, & S_2 = 2^{-2}, & S_3 = 1, & S_4 = 2^{-4}, & S_5 = 1, & \ldots \\ T_1 = 2^{-1}, & T_2 = 1, & T_3 = 2^{-3}, & T_4 = 1, & T_5 = 2^{-5}, & \ldots \end{array} \tag{2}$$

because $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} S_i$ and $\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} T_i$ are clearly unbounded while $\sum_{i=1}^{\infty} \min\{a_i, b_i\} = \sum_{i=1}^{\infty} \min\{S_i, T_i\} = \sum_{i=1}^{\infty} 2^{-i} = 1$ is bounded.

Constructions satisfying (2) revolve around choosing large values of k_i to ensure the monotonicities of a_i and b_i. Here is an example:

i	1	2	3	4	5	$\cdots$
$k_{i+1}-k_i$	1	2^1	2^{1+2}	2^{1+2+3}	$2^{1+2+3+4}$	$\cdots$
a_{k_i}	1	2^{-1-2}	2^{-1-2}	$2^{-1-2-3-4}$	$2^{-1-2-3-4}$	$\cdots$
b_{k_i}	2^{-1}	2^{-1}	2^{-1-2-3}	2^{-1-2-3}	$2^{-1-2-3-4-5}$	$\cdots$

Equations (2) can be easily checked to hold using (1), completing the solution.

Bonus: Assume there exists a sequence of a_i that, along with $b_i = 1/i$, satisfies the required conditions. If $b_i \le a_i$ occurs only finitely many times, then for large enough i, we always have $\min\{a_i, b_i\} = a_i$ and the minimum sequence has to be unbounded. That is a contradiction. Hence there exists an infinite sequence of indices $\{t_n, n \ge 1\}$ such that $b_{t_n} \le a_{t_n}$. For convenience let $t_0 = 0$.

Write the partial sums of the minimum sequence as

$$\begin{aligned} U_n &:= \sum_{i=t_{n-1}+1}^{t_n} \min\{a_i, b_i\} \\ &\ge (t_n - t_{n-1}) \min\{a_{t_n}, b_{t_n}\} = \frac{t_n - t_{n-1}}{t_n} = 1 - \frac{t_{n-1}}{t_n} \\ \implies \quad \frac{t_{n-1}}{t_n} &\ge 1 - U_n. \qquad (3) \end{aligned}$$

Since $\sum_{n=1}^{\infty} U_n = \sum_{i=1}^{\infty} \min\{a_i, b_i\} < \infty$, we must have $U_n \to 0$ as $n \to \infty$, and both sides of (3) are eventually positive. Now compare U_n with the partial sums of the harmonic series, for large enough n,

$$\sum_{i=t_{n-1}+1}^{t_n} \frac{1}{i} < \int_{t_{n-1}}^{t_n} \frac{1}{x}\, \mathrm{d}x = -\ln \frac{t_{n-1}}{t_n} \le -\ln(1 - U_n).$$

By L'Hôpital's rule,

$$\lim_{n\to\infty} \frac{-\ln(1-U_n)}{U_n} = \lim_{x\to 0} \frac{-\ln(1-x)}{x} = \lim_{x\to 0} \frac{1}{1-x} = 1.$$

Thus for large enough n,

$$\sum_{i=t_{n-1}+1}^{t_n} \frac{1}{i} < -\ln(1-U_n) < 2U_n.$$

But since the harmonic series is unbounded, the sum of U_n must also be unbounded. This is a contradiction, and therefore no such sequences exist.

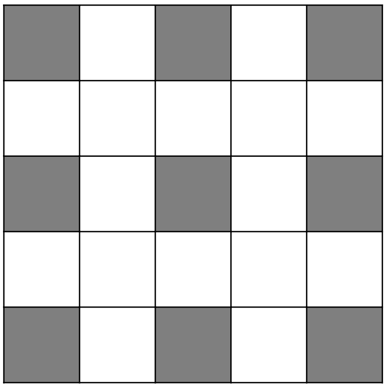
Tales of tiles

1. *An L-tromino is a 2×2 tile with one unit square removed. Tila has tiled a 5×5 square with eight L-trominoes and a single 1×1 square tile. Where can the 1×1 tile possibly be?*

2. *Tyler is tiling his rectangular bathroom with some* 2×2 *square tiles and some* 4×1 *rectangular tiles. After arranging the tiles to cover the bathroom perfectly without overlap, the clumsy Tyler accidentally smashes one of his tiles. Unfortunately, the only spare tile is of the other shape to the one he smashed. Will Tyler be able to rearrange the remaining unsmashed tiles to perfectly cover his bathroom again?*
3. *For which integers* n*, does there exist a shape which can be tiled using* 2×1 *dominoes in exactly* n *different ways?*

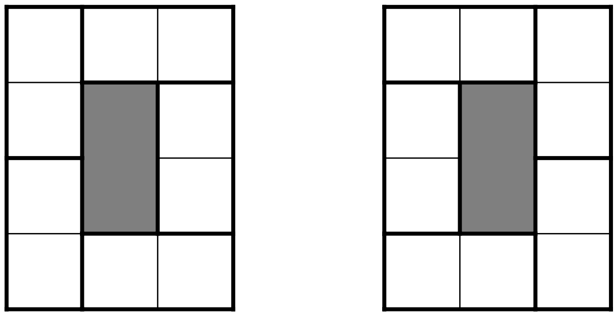
Solution by Jensen Lai:

1. Colour nine of the squares as follows.

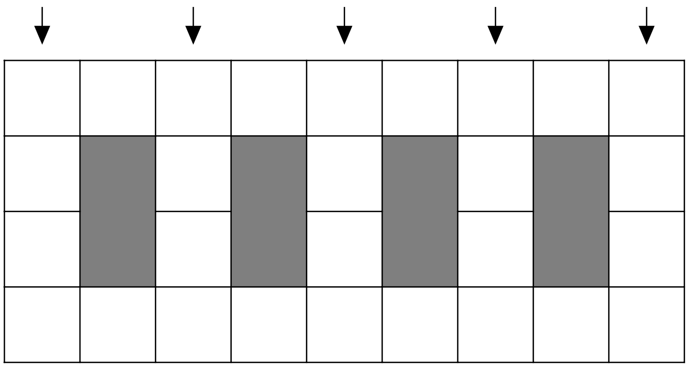


   Each L-tromino can cover at most one of the coloured squares. Since there are only eight L-trominoes, the 1×1 square tile must also cover a coloured square. It turns out that all nine positions are possible. The constructions are quite straightforward and will be left to the reader.
2. Use the same colouring as part 1, that is, colour the squares with both odd column and odd row numbers. Each 2×2 tile covers exactly one (an odd number) coloured square, while each 4×1 tile covers either two or zero (an even number) coloured squares. Hence the parity of the number of coloured squares is the same as the parity of the number of 2×2 tiles used.

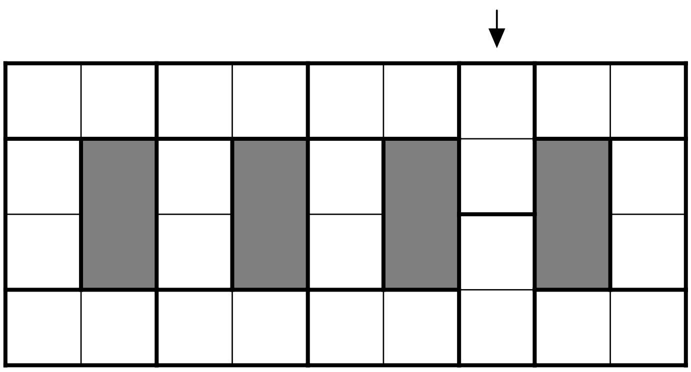
   The replacement of the smashed tile also changes the parity of the number of 2×2 tiles. Therefore the bathroom can no longer be tiled perfectly.
3. For $n = 1$, a 2×1 rectangle can be tiled by a domino in exactly one way. For $n = 2$, the following shape can be tiled in exactly two ways. Note that there is a 2×1 rectangular hole in the middle which is not part of the shape.



This construction can actually be generalised to any n. For $n = 5$ we have the following shape (with four holes).

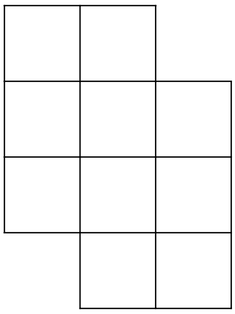


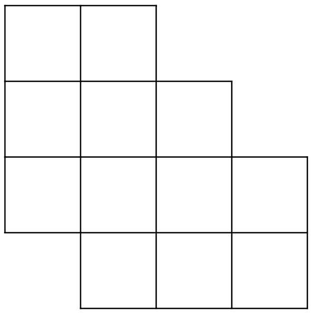
Call a column *strong* if it has two vertical dominoes. Out of the five columns indicated, at least one must be strong. But as soon as we choose a strong column, the tiling is forced and no other columns can also be strong. For example, if the fourth column is strong, the following tiling is forced.



Since there were five possible choices of strong columns, there are exactly five ways to tile the shape. By similar arguments, for all positive integers n, there exists a shape which can be tiled by dominoes in exactly n ways.

Comment: There is actually an even trickier construction which only uses $2n$ squares and has no holes. The following shapes are for $n = 5$ and $n = 6$. See if you can figure out how it works exactly.





Switching sequence

There is an infinite number of rooms in a row, numbered $1, 2, \ldots$*, and each has a lightbulb. The switch to the room k lightbulb is in room* $k-1$ *for* $k \geq 2$ *and the room 1 lightbulb has no switch. Furthermore a switch can only be used if the light is on in its residing room. If more than n lightbulbs are on at the same time, the power supply will overheat and explode. Initially, only the room 1 lightbulb is on. What is the largest numbered room you can illuminate without causing a catastrophe?*

Solution by Andrew Elvey Price: Say the room 1 lightbulb uses no power since its always on, while every other lightbulb uses one watt. Then the largest numbered room which can be illuminated using $n - 1 = k$ watts is room 2^k. The case for $k = 1$ is easily checked.

Note that every switching algorithm is reversible if the available power does not change. Proceed by induction, assume the following statements are true when k watts are available.

- A_k: Ignoring room 1, the leftmost illuminated room number cannot be greater than $2^{k-1} + 1$;
- B_k: It is not possible to illuminate rooms greater than 2^k;
- C_k: It is possible to illuminate room 2^k.

Now we prove the corresponding statements for $k + 1$ watts.

A_{k+1}: Assume the contrary. Say we can reach a state T where the leftmost illuminated room, apart from room 1, has number $s > 2^k + 1$. On the way to T, let the last moment when the room s light is switched on be state U. By definition, room $s - 1$ is also illuminated at that time. Consider the algorithm from state U to state T. There is one less watt available since the room s light is always on, but it is still possible to turn off the room $s - 1$ light. Since the moves are reversible, there must exist an algorithm to illuminate room $k - 1 > 2^{n-1}$ from the beginning using only k watts, contradicting A_k.

B_{k+1}: By A_{k+1}, the lights in rooms $2, 3, \ldots, 2^k - 1, 2^k$ must use up at least one watt at all times. So within any algorithm, the set of switching moves performed on rooms greater than 2^k is also an algorithm using only k watts. Applying B_k, it is not possible to illuminate rooms greater than $2^k + 2^k = 2^{k+1}$.

C_{k+1}: Use C_k to light up room 2^k using only k watts, then switch on the room $2^k + 1$ lightbulb using the last watt. Now reverse C_k to turn off everything but rooms 1 and $2^k + 1$. Finally, apply C_k again, but starting from room $2^k + 1$. This will illuminate room $(2^k + 1) + (2^k) - 1 = 2^{k+1}$, as required.

This completes the induction. Therefore the largest numbered room we can illuminate is room $2^k = 2^{n-1}$.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.