Graphs and C^*-algebras

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Abstract

We outline Adam Sørensen's recent characterisation of stable isomorphism of simple unital graph C^*-algebras in terms of moves on graphs. Along the way we touch on some fascinating connections between combinatorics, symbolic dynamics and operator algebras.

1. Introduction

Since the late 1990s there has been widespread interest in *graph C^*-algebras*. These arose from Cuntz and Krieger's work in the early eighties [5], [6] and were first investigated by Enomoto and Watatani [7], but took off subsequent to work by Kumjian, Pask, Raeburn and Renault in the late 1990s [9], [10]. An isomorphism of graphs determines an isomorphism of the associated C^*-algebras, but not conversely: many non-isomorphic graphs typically give rise to the same C^*-algebra. In this note, we discuss recent results of Adam Sørensen [13] which describe the equivalence relation on directed graphs which corresponds to stable isomorphism of simple unital graph C^*-algebras.

2. Graphs and C^*-algebras

In this note we will consider a directed graph to be a quadruple $E = (E^0, E^1, r, s)$ where E^0 and E^1 are finite sets and r and s are functions from E^1 to E^0. We think of the elements of E^0 as the vertices (and picture them as dots) and the elements of E^1 as the edges (and picture them as arrows connecting one dot to another). The maps r and s indicate the directions of the edges: each edge $e \in E^1$ points from $s(e)$ to $r(e)$.

We require $|E^0| < \infty$, but make no other restrictions. In particular, we allow loops and parallel edges, and though E^0 is required to be finite, E^1 is not.

We are interested in *C^*-algebras* associated to graphs. C^*-algebras arose in the early 20th century as models for quantum mechanics. They have an abstract axiomatic description, but a deep theorem of Gelfand and Naimark in the 1940s says

Invited technical paper, communicated by Mathai Varghese.

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that, up to isomorphism, C^*-algebras are precisely the closed *-subalgebras of the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on Hilbert space. So C^*-algebras are like infinite-dimensional analogs of multi-matrix algebras. Two C^*-algebras are *stably isomorphic* if they are isomorphic after tensoring with the C^*-algebra of compact operators on Hilbert space. For separable C^*-algebras, stable isomorphism is equivalent to the appropriate notion of Morita equivalence for C^*-algebras, and it preserves many important C^*-algebraic properties. A C^*-algebra A is said to be *simple* if every nonzero *-preserving homomorphism of A is injective.

If $E = (E^0, E^1, r, s)$ is a directed graph, we construct a C^*-algebra from E by associating operators on a Hilbert space $\mathcal{H}$ to the vertices and to the edges. We associate to the vertices v, projections P_v onto mutually orthogonal subspaces $\mathcal{H}_v$ of $\mathcal{H}$, and to each edge e an operator S_e which carries $\mathcal{H}_{s(e)}$ isometrically onto a subspace of $\mathcal{H}_{r(e)}$ and vanishes on $\mathcal{H}_v$ for $v \neq s(e)$. We insist that the subspaces $S_e\mathcal{H}$, $e \in E^1$ are mutually orthogonal, and that whenever $r^{-1}(v)$ is finite and nonempty, the subspaces $S_e\mathcal{H}$ such that $r(e) = v$ span $\mathcal{H}_v$. Such a collection of operators $\{P_v, S_e \colon v \in E^0,\ e \in E^1\}$ is called a *Cuntz–Krieger E-family*. Each Cuntz–Krieger E-family generates a C^*-algebra, namely the closure in $\mathcal{B}(\mathcal{H})$ of the linear span of all finite products of the P_v, the S_e, and their adjoints.

Every directed graph admits a Cuntz–Krieger family in which all the $\mathcal{H}_v$ are nontrivial. Moreover, the Cuntz–Krieger uniqueness theorem [1], [10] for graph algebras implies that a graph E satisfies the conditions

(1) given any v, $w \in E^0$ such that $\{e \in E^1 \colon r(e) = v\}$ is either empty or infinite, there is a path in E from v to w, and
(2) on every cycle in E there is at least one vertex v such that $|r^{-1}(v)| > 1$

if and only if every pair of nonzero Cuntz–Krieger E-families generate canonically isomorphic C^*-algebras. It then makes sense to describe the C^*-algebra generated by any Cuntz–Krieger E-family as *the* C^*-algebra of E, and denote it by $C^*(E)$. The Cuntz–Krieger uniqueness theorem implies that $C^*(E)$ is simple.

If E and F are directed graphs satisfying (1) and (2) and ϕ is an isomorphism of E onto F, then for any Cuntz–Krieger F-family $\{Q_w, T_f \colon w \in F^0,\ f \in F^1\}$ the assignments $P_v := Q_{\phi(v)}$ and $S_e := T_{\phi(e)}$ determine a Cuntz–Krieger E-family which generates $C^*(F)$. So the Cuntz–Krieger uniqueness theorem implies that $C^*(E)$ is isomorphic to $C^*(F)$. However, it is usually not true that if $C^*(E)$ and $C^*(F)$ are isomorphic, then E and F are isomorphic also. One is led to ask: what is the equivalence relation $\sim$ on directed graphs satisfying (1) and (2) such that $E \sim F$ if and only if $C^*(E) \cong C^*(F)$? Or the related — and, as it turns out, more tractable — question: what equivalence relation on graphs corresponds to stable isomorphism of $C^*(E)$ and $C^*(F)$?

3. Flow equivalence and moves

There are two approaches to answering the question posed in the preceding section. One is to use classification results for C^*-algebras. Associated to each C^*-algebra A are two groups: $K_0(A)$ and $K_1(A)$. Roughly speaking, $K_0(A)$ contains information about the projections in A, and $K_1(A)$ contains information about unitaries in A. Over the last 30 years or so a concerted classification program instigated and championed by Elliott has established that many classes of C^*-algebras are classified by their K-groups: two C^*-algebras in one of these classes are stably isomorphic if and only if their K-groups are isomorphic. In particular, simple graph C^*-algebras are classified up to stable isomorphism by their K-groups, and $K_0(C^*(E))$ and $K_1(C^*(E))$ can be read off E. The drawback of this approach is that the classification results involved frequently assert the existence of a stable isomorphism without any indication of an explicit formula.

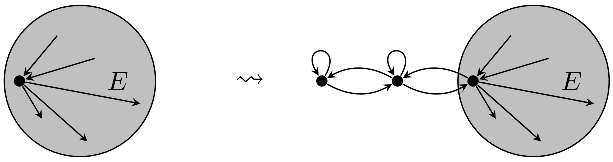
Sørensen's rather different approach is inspired by work of Parry and Sullivan and subsequent work by Franks on classification of irreducible subshifts up to flow equivalence. Let $\mathcal{A}$ be a set endowed with the discrete topology. Write $\mathcal{A}^{\mathbb{Z}}$ for the collection of all functions from $\mathbb{Z}$ to $\mathcal{A}$. Define $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by $\sigma(x)_n = x_{n+1}$. A *subshift over* $\mathcal{A}$ is a subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ which is topologically closed with respect to pointwise convergence and is invariant for σ and σ^{-1}. Each directed graph E determines a subshift X_E as follows: $X_E = \{x \in (E^1)^{\mathbb{Z}}\colon s(x_n) = r(x_{n+1}) \text{ for all } n \in \mathbb{Z}\}$.

Flow equivalence of subshifts is a fairly weak form of equivalence which preserves many key properties. Parry and Sullivan [11] proved that flow equivalence is generated as an equivalence relation by *conjugacy* (the appropriate notion of isomorphism of shift spaces) and *symbol expansion*, which replaces a shift X over $\mathcal{A}$ with the shift over $\mathcal{A}\sqcup\{\star\}$ obtained by replacing every instance of some fixed $a \in \mathcal{A}$ with the string $a\star$. They also described two moves on finite directed graphs (that is E^0 and E^1 are both finite) which generate the equivalence relation corresponding to flow-equivalence of irreducible subshifts, thus answering a question of Bowen [3].

Given a finite graph E, let A_E denote the $E^0 \times E^0$ integer-valued matrix such that $A_E(v,w) = |\{e \in E^1\colon r(e) = v \text{ and } s(e) = w\}|$. We regard $I - A_E$ as a homomorphism of $\mathbb{Z}^{E^0}$, so that its cokernel is $\mathbb{Z}^{E^0}/(I - A_E)\mathbb{Z}^{E^0}$. Using Parry and Sullivan's results, Franks [8] proved that if E and F are strongly-connected finite directed graphs, then X_E and X_F are flow equivalent if and only if

$$\operatorname{coker}(1 - A_E) \cong \operatorname{coker}(1 - A_F) \quad \text{and} \quad \operatorname{sgn}(\det(1 - A_E)) = \operatorname{sgn}(\det(1 - A_F)).$$

Cuntz and Krieger's work [6] established a remarkable relationship between C^*-algebras and the theory of subshifts. It was discovered early on that if E and F are finite directed graphs satisfying conditions (1) and (2), and if X_E and X_F are flow equivalent, then $C^*(E)$ and $C^*(F)$ are stably isomorphic. It was unknown whether the converse held, until Rørdam [12] proved that $C^*(E)$ and $C^*(F)$ are stably isomorphic if and only if E can be transformed into F using a finite sequence of the moves introduced by Parry and Sullivan and the *Cuntz splice* (illustrated below)



applied at a vertex v supporting at least two first-return paths (directed cycles which visit v only at their ranges and sources). The Cuntz-splice reverses the sign of $\det(1 - A_E)$ but preserves $\operatorname{coker}(1 - A_E)$, so $C^*(E)$ and $C^*(F)$ are stably isomorphic if and only if $\operatorname{coker}(1 - A_E) \cong \operatorname{coker}(1 - A_F)$.

4. Sørensen's result

Rørdam's result was obtained using a K-theoretic approach. Sørensen recently returned to considering a direct approach to moves on graphs which generate the equivalence relation corresponding to stable isomorphism of simple unital C^*-algebras. He listed the following five basic moves:

- (`S`) remove a vertex v such that $s^{-1}(v)$ is empty
- (`R`) collapse an edge e to a point if $r^{-1}(r(e)) = \{e\} = s^{-1}(s(e))$
- (`I`) distribute the edges entering a vertex v with $0 < |r^{-1}(v)| < \infty$ amongst n new vertices $v_1, \dots, v_n$ (for each edge leaving v add a copy leaving each v_i)
- (`O`) distribute the edges leaving a vertex v with $|r^{-1}(v)| < \infty$ and $s^{-1}(v) \neq \emptyset$ amongst n new vertices $v_1, \dots, v_n$ (for each edge entering v add a copy entering each v_i)
- (`C`) perform a Cuntz-splice at a vertex supporting two distinct first-return paths.

In the context of graph C^*-algebras, the moves `I` and `O` arose in work of Bates and Pask [2], and the moves `S` and `R` are special cases of a fairly complicated construction due to Crisp and Gow [4]; but all of these were themselves inspired by the connection with symbolic dynamics and flow-equivalence discussed above.

Theorem 4.1 (Sørensen, 2011)**.** *Let E and F be directed graphs with finitely many vertices satisfying conditions (1) and (2). If E contains at least one vertex v such that $r^{-1}(v)$ is either empty or infinite, then $C^*(E)$ and $C^*(F)$ are stably isomorphic if and only if E can be transformed into F by a finite sequence of moves `S`, `R`, `I` and `O`. If E contains no such vertex then $C^*(E)$ is stably isomorphic to $C^*(F)$ if and only if E can be transformed into F by a finite sequence of moves `S`, `R`, `I`, `O` and `C`.*

What is particularly appealing about this result is Sørensen's proof in the case when there is an infinite receiver present. It is an open (and perhaps impossible) challenge to find a constructive proof that the Cuntz-splice preserves the stable isomorphism class of a graph C^*-algebra. But the stable isomorphisms corresponding to moves `S`, `R`, `O` and `I` are quite concrete: given a subset V of E^0, the element $P_V := \sum_{v \in V} P_v$ is a projection in $C^*(E)$. Results by Crisp and Gow [4] and

Bates and Pask [2] show that if F is obtained from E by one of the moves `S`, `R`, `O` and `I`, then there are subsets $V \subseteq E^0$ and $W \subseteq F^0$ and an isomorphism $P_V C^*(E) P_V \cong P_W C^*(F) P_W$ that carries generators to generators. This gives a concrete stable isomorphism between $C^*(E)$ and $C^*(F)$ via Morita-equivalence theory for C^*-algebras and so we obtain the following corollary.

Corollary 4.1. Let E and F be directed graphs each satisfying conditions (1) and (2) and each containing at least one infinite receiver. Then $C^*(E)$ and $C^*(F)$ are stably isomorphic if and only if there is a finite sequence of graphs $E_0, \dots E_n$ and subsets $V_i \subseteq E_i^0$ and $W_i \in E_{i+1}^0$ for $0 \le i < n$ such that $E_0 = E$, $E_n = F$, and for each $i < n$ a concrete isomorphism $P_{V_i} C^*(E_i) P_{V_i} \cong P_{W_i} C^*(E_{i+1}) P_{W_i}$.

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