Legendre polynomials and series for $1/\pi$

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One of Ramanujan's surprising discoveries [5] is series of the form

$$\sum_{n=0}^{\infty} h_n(a_0 n + b_0) z_0^n = \frac{1}{\pi}, \tag{1}$$

where h_n is a *hypergeometric* term, that is h_{n+1}/h_n is a rational function of n. More amazingly, some series contain rational summands, for example,

$$\sum_{n=0}^{\infty} \binom{2n}{n}^3 \left(2n + \frac{1}{2}\right)(-2^{-6})^n = 1/\pi.$$

In the language of hypergeometric series, (1) says a linear combination of a suitable ${}_3F_2$ and its derivative near 0 gives $1/\pi$. Research into (1) continues, one aim being to replace h_n by more general sequences.

Proofs for (1) are just as fascinating as the series themselves, involving mainly three different fields, all of which are used in our work:

- Hypergeometric series: special ${}_2F_1$s, such as the elliptic integral K, can be evaluated in closed form at certain algebraic arguments, and whose square is a ${}_3F_2$ via *Clausen's identity*;
- Modular forms: parametrise components in (1) (such as z_0) in terms of modular forms, and observe that modular forms of the same weight are related by algebraic equations;
- Experimental mathematics: empirical discovery by integer relation programs like PSLQ, and automatic proof and identity generation using Celine's and Zeilberger's algorithms.

In 2002, T. Sato extended h_n to include *Apéry-like sequences* u_n which subsume the hypergeometric terms, and which satisfy the recursion

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \qquad u_{-1} = 0, \qquad u_0 = 1.$$

It is believed that there are 14 triplets $(a, b, c) \in \mathbb{Z}^3$ that produce integer u_n for all n, and these sequences are studied for their arithmetic properties, for example, Apéry used them in his irrationality proofs of $\zeta(2)$ and $\zeta(3)$. Subsequent work of H.H. Chan, W. Zudilin *et al.* [2] extended h_n to the product of a hypergeometric term and an Apéry-like sequence. More recently, Z.W. Sun conjectured [6] that h_n may also be a hypergeometric term times $P_{\alpha n}(x_0)$ for $\alpha \in \{1, 2, 3\}$; here $P_n(x)$ denotes the *Legendre polynomial.*

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The all-important Clausen's identity follows from Bailey's identity for a particular Appell's hypergeometric function. In the 1950s, F. Brafman used Bailey's identity to prove some unusual (but nearly forgotten) generating functions for Legendre polynomials [1], which may be manipulated via modular and hypergeometric machinery [3] to validate Sun's observations for $\alpha = 1$; moreover, in the same work we provide a recipe for producing such series at will.

In our proof of the $\alpha = 2$ and $\alpha = 3$ cases, we successfully generalised Bailey's identity to encompass all Apéry-like sequences, and produced the very general generating function [4]:

$$\sum_{n=0}^{\infty} u_n x^n \sum_{n=0}^{\infty} u_n y^n = \frac{1}{1-cxy} \sum_{n=0}^{\infty} u_n \sum_{m=0}^{n} \binom{n}{m}^2 g(x,y)^m g(y,x)^{n-m}, \qquad (2)$$

where $g(x,y) = x(1-ay+cy^2)/(1-cxy)^2$. Notably, (2) was discovered, and then proven, *experimentally* with significant computer input as compared to the need for human intervention (the key step in the proof involves using a computer to find a partial differential equation which annihilates both sides). When used in conjunction with a theorem of H.M. Srivastava, (2) gives new connections between Legendre polynomials (and other special functions) and Apéry-like sequences, while at the same time offers the most general theory of series for $1/\pi$. In particular, it allows h_n to be an Apéry-like sequence multiplied by a Legendre polynomial. An example of this, with rational summands, is

$$\sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \left(\frac{-1}{8}\right)^k \binom{k}{j}^3 \right] P_n\left(\frac{5}{3\sqrt{3}}\right) n \left(\frac{4}{3\sqrt{3}}\right)^n = \frac{9\sqrt{3}}{2\pi}.$$

References

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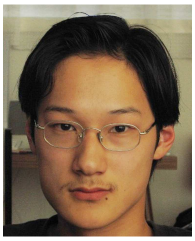

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