A short introduction to self-similar groups—Solutions

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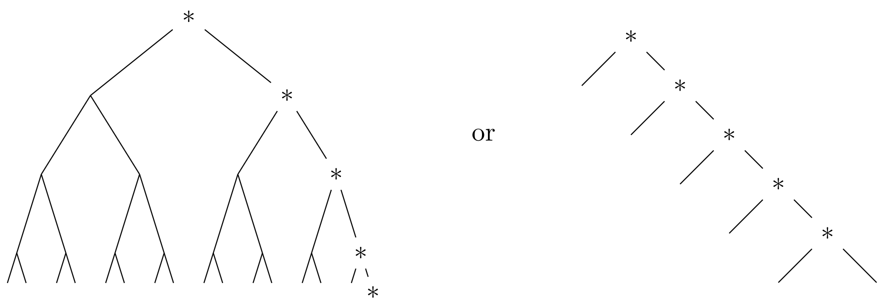
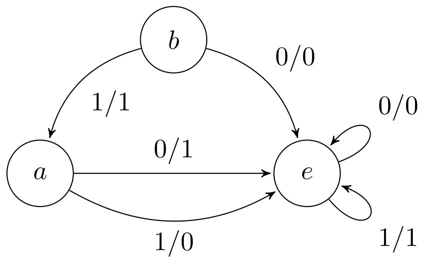
Exercise 2.1. Draw the portrait of the automorphism x in this example.



Exercise 2.2. Draw the automaton encoding the rules for a and b in the previous section.



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Exercise 2.3. If you know some basic group theory, do you recognise the group generated by $\{x\}$ and the group generated by $\{a, b\}$?

The first group consists of powers of x. The only relation it could satisfy is $x^n = e$ for some n which we may as well assume to be positive. Consider the node labeled $000\cdots 0$ on the left side of T. Applying x n times sends it to $100\cdots 0 \to 010\cdots 0 \to 110\cdots 0 \to 001\cdots 0$ which is like *adding* 1 to a binary number (written backwards), so for any positive n the node will never get back to $000\cdots 0$, so there are no relations, and the group is isomorphic to $\mathbb{Z}$.

For the second group—$a^2 = e$, and $b^2 = e$, so we have some kind of dihedral group. The element ab sends $11w \to 01w \to 10w \to 00w \to 11w$ so you can see that this group only acts transitively on the first two levels. This also shows us that $(ab)^4 = e$, so this group is the dihedral group or order 8.

Exercise 3.1. Write the self-similar rules for a, b, c, d by reading off the automation.

$$\begin{aligned} a(0w) &= 1e(w), & a(1w) &= 0e(w),\\ b(0w) &= 0a(w), & b(1w) &= 1c(w),\\ c(0w) &= 0a(w), & c(1w) &= 1d(w),\\ d(0w) &= 0e(w), & d(1w) &= 1b(w). \end{aligned}$$

Exercise 3.2. Using the portraits, or otherwise, work out what bb, cc, dd and bc do. Which automorphisms are they the same as?

Applying b, or d twice we switch the subtrees at the $*$ed nodes twice, so $b^2 = c^2 = d^2 = e$. On the other hand, applying b then c, we switch the nodes labeled $0, 10, 110, 1110, \ldots$ as follows: $(1)^{3k}0$ nodes get switched twice, while $(1)^{3k}10$ get switched just once (by b) and $(1)^{3k}110$ get switched just once (by c), so bc is the same automorphism as d.

Note that since $bc = d$ multiplying both sides on the right by d we get $bcd = e$. In fact you can show that the subgroup generated by b, c, d is the Klein four group (the noncyclic group of order 4).

Exercise 3.3. Show that $adad$ is the same automorphism as $dada$.

$adad(0w) = dad(1.w) = ad(1.b(w)) = d(0.b(w)) = 0.b(w)$ since $d(1w) = 1.b(w)$, and $dada(0w) = ada(0w) = da(1.w) = a(1.b(w)) = 0.b(w)$

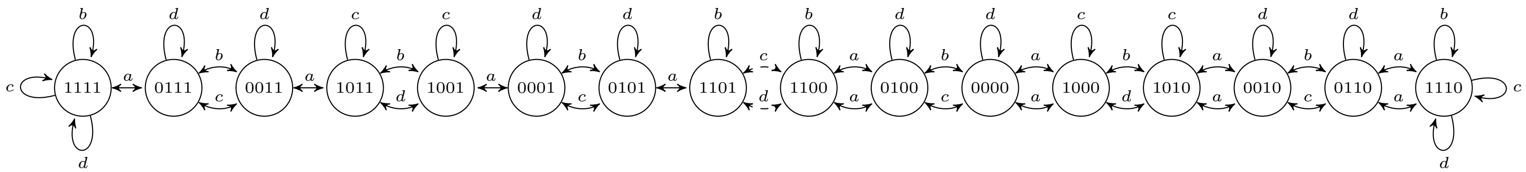
and $ada(1w) = da(0.w) = a(0.w) = 1.w$ (that is, acts like the identity on $1w$ nodes, i.e. the right subtree) so $adad(1w) = d(1.w) = 1.b(w)$ and $dada(1w) = ada(1.b(w)) = 1.b(w)$.

You could also apply the word problem algorithm to $adadadad = (ada)d(ada)d$ which we rewrite as $(b)e(b)e$ which is $b^2 = e$, and (using the next exercise) $(e)b(e)b$ which is also e.

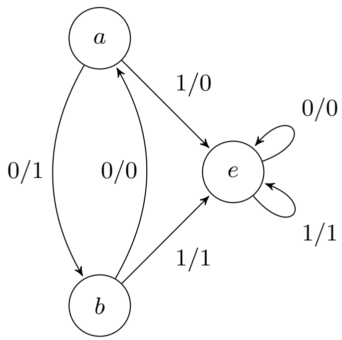
Exercise 4.1. Work out the replacement rules for b, c, d, aba, aca, ada for the right subtree.

b acts like c, c acts like d, and d acts like b, while aba acts like c, aca acts like d, and ada acts like e.

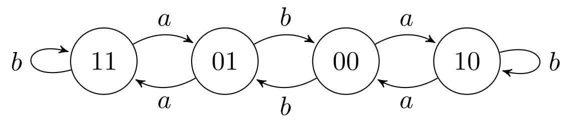
Exercise 6.1. Draw the Schreier graph of the action of B for levels $4, 5, \ldots$ of T. What do you see?



Exercise 8.1. Draw an automaton (with states labelled a, b, e) which encodes these rules.

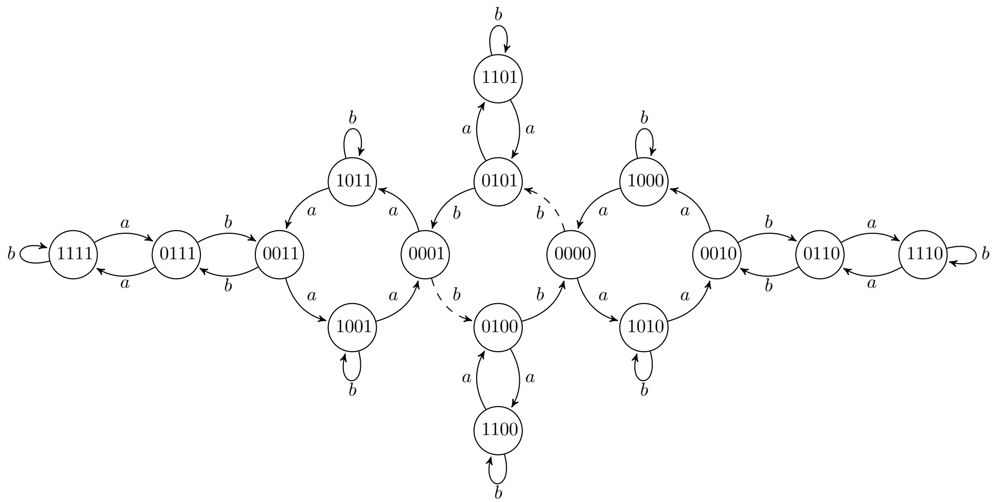


Exercise 8.2. Draw the Schreier graph of the action of B on level 2 of T.



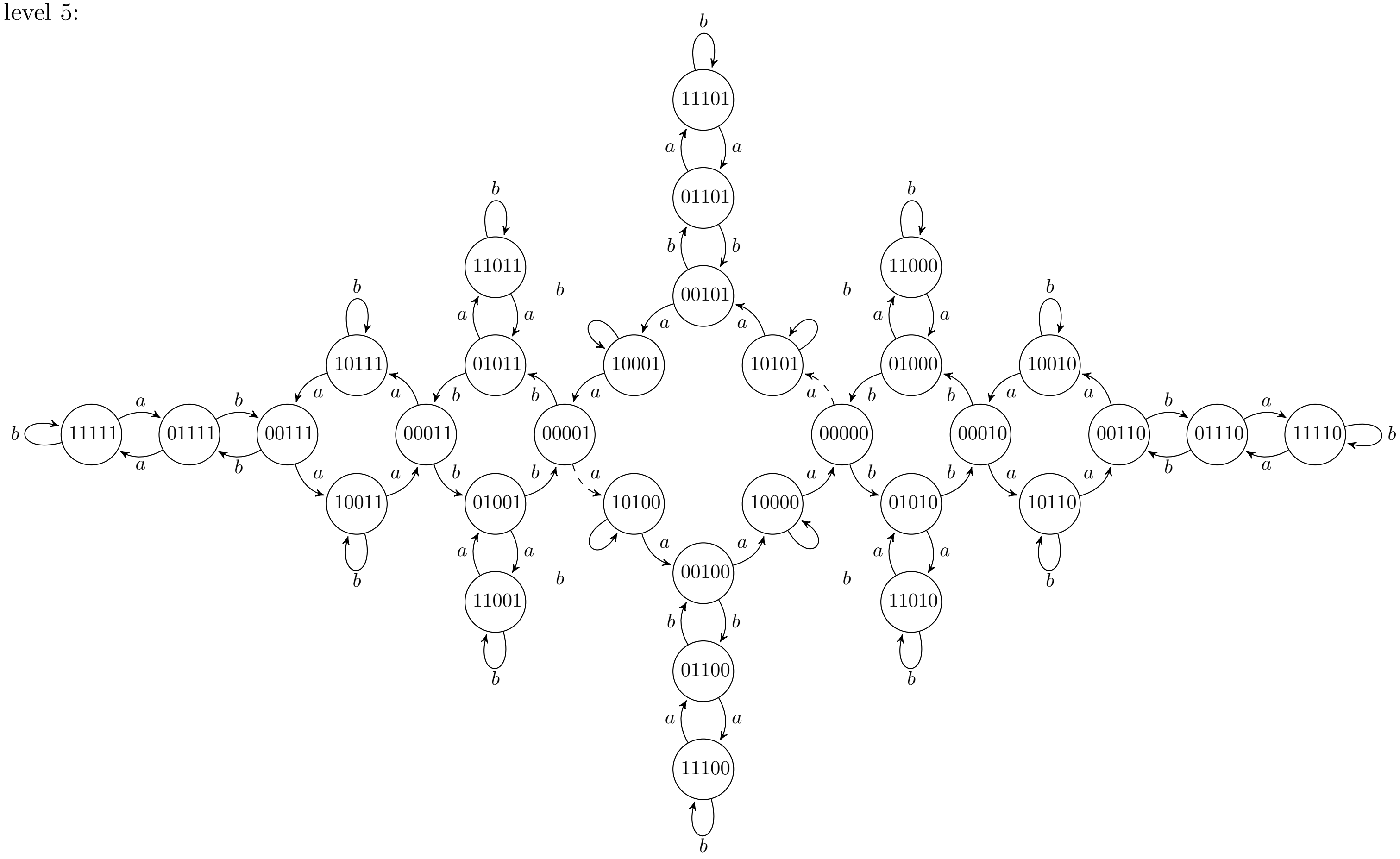
Exercise 8.3. Draw the Schreier graph of the action of B for levels $4, 5, \ldots$ of T. What do you see?

level 4:



Cut and connect the dashed b edges to get two copies of the graph for level 3.

level 5:



As you go further down the levels of T, the Schreier graphs limit to a fractal picture, which looks a bit like the Basilica San Marco in Venice. The connection with fractals (the limit is in fact the Julia set of $1-z^2$) is explored in detail in Nekrashevich's book. The basilica group is the first example of a group that is amenable but not subexponentially amenable.