Puzzle Corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner No. 28. Each Puzzle Corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes, SA 5095.

The deadline for submission of solutions for Puzzle Corner 28 is 30 September 2012. The solutions to Puzzle Corner 28 will appear in Puzzle Corner 30 in the November 2012 issue of the *Gazette*.

Notice: If you have heard of, read, or created any interesting mathematical puzzles that you feel are worthy of being included in the Puzzle Corner, I would love to hear from you! They don't have to be difficult or sophisticated. Your submissions may very well be featured in a future Puzzle Corner, testing the wits of other avid readers.

Mean marks

The final exam marks have just been released. Within each class, the boys have a higher average score than the girls. Does that necessarily mean the boys have a higher average score across the entire grade?

Enlarged enclosure

You have four straight pieces of fencing, that are 1, 4, 7 and 8 metres in length. What is the greatest area you can enclose with these pieces?

Photo: Magda18 (SXC)

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Flipping fun

There is a coin at every integer point of the number line. A stencil with a finite set of fixed holes at integer distances is chosen. The stencil may be moved along the number line, and for any fixed position of the stencil, one may simultaneously flip all the coins accessible through the holes. Initially all coins are showing heads. Prove that for any stencil it is possible to get exactly two tails after a finite number of such operations.

Mad hat party

Photo: Nathalie Dulex

The Mad Hatter is holding a hat party, where every guest must bring his or her own hat. At the party, whenever two guests greet each other, they have to swap their hats. In order to save time, each pair of guests is only allowed to greet each other at most once.

After a plethora of greetings, the Mad Hatter notices that it is no longer possible to return all hats to their respective owners through more greetings. To sensibly resolve this maddening conundrum, he decides to bring in even more hat wearing guests, to allow for even more greetings and hat swappings. How many extra guests are needed to return all hats (including the extra ones) to their rightful owners?

Chord variations

There are 2012 points on the circumference of a circle, dividing it into 2012 equal arcs. The points are to be labelled with $A_1, \ldots, A_{2012}$ in some order.

1. Can you label the points in a way so that no two of the 2012 chords

$$A_1A_2\ ,\ A_2A_3\ ,\ \ldots\ ,\ A_{2011}A_{2012}\ ,\ A_{2012}A_1$$

   are parallel?

2. Can you label the points in a way so that no two of the 1006 chords

$$A_1A_2\ ,\ A_3A_4\ ,\ \ldots\ ,\ A_{2009}A_{2010}\ ,\ A_{2011}A_{2012}$$

   have equal length?

Solutions to Puzzle Corner 26

Many thanks to everyone who submitted solutions. The \$50 book voucher for the best submission to Puzzle Corner 26 is awarded to Adrian Nelson. Congratulations!

Ratio of radii

A sphere with radius r is inscribed in a regular tetrahedron, which is inscribed in a larger sphere with radius R. Find the ratio of R to r.

Solution by Alan Jones: Let the vertices of the tetrahedron be V_1, V_2, V_3 and V_4. Let the centre of the spheres be the origin O. By symmetry, O is also the centroid of the tetrahedron, in other words:

$$\frac{\overrightarrow{OV_1}+\overrightarrow{OV_2}+\overrightarrow{OV_3}+\overrightarrow{OV_4}}{4}=\overrightarrow{OO}=0 \implies -\overrightarrow{OV_1}=\overrightarrow{OV_2}+\overrightarrow{OV_3}+\overrightarrow{OV_4}.$$

Now, R is the distance from O to V_1, while r is the distance from O to the centroid of the triangle $V_2V_3V_4$. Therefore

$$\frac{R}{r}=\frac{\left|\overrightarrow{OV_1}\right|}{\left|\frac{\overrightarrow{OV_2}+\overrightarrow{OV_3}+\overrightarrow{OV_4}}{3}\right|}=\frac{\left|\overrightarrow{OV_1}\right|}{\left|\frac{-\overrightarrow{OV_1}}{3}\right|}=3.$$

Counterfeit coins

There are 20 coins in front of you, two of which are counterfeits. The genuine coins are identical in weight, but the counterfeits are slightly lighter. Can you identify 10 genuine coins by using a balance-scale twice? Note that the two counterfeits are not necessarily the same weight as each other.

Solution by John van der Hoek: First note a couple of facts when weighing two piles of equal size:

- If the scale is balanced, then either both piles are genuine, or both contain counterfeit coins;
- If the scale is not balanced, then the lighter pile must contain at least one counterfeit coin.

Now divide the 20 coins into four piles of five, labelled A, B, C and D. First weigh A against B.

If $A < B$, then weigh C against D. If $C < D$, then B and D are genuine. If $C > D$, then B and C are genuine. If $C = D$, then C and D are genuine, since we cannot have counterfeit coins in piles A, C and D simultaneously.

If $B < A$, the situation is similar to $A < B$.

If $A = B$, then weigh $A + B$ against $C + D$. Recall that since $A + B$ must contain either zero or two counterfeit coins, $A+B = C+D$ cannot occur. If $A+B < C+D$, then piles $A + B$ must contain two counterfeit coins, so C and D are genuine. If $A + B > C + D$, then A and B must both be genuine.

In all cases, we have found two genuine piles, as required.

Poor turnout

Following the service, the vicar asked the bell-ringer if he could work out the ages of the three people who attended today, given that the product of their ages was 2450 and the sum was twice the age of the bell-ringer. After some thought, the bell-ringer was unable to do so. The vicar then revealed that he (the vicar) was in fact older than all three of them. Upon hearing that, the bell-ringer quickly responded with the three ages. What were the ages of all five people?

Solution by John Giles: Let the ages of the three attendees be x, y and z, the age of the bell-ringer be b and the age of the vicar be v. Then

$$xyz = 2450 = 2 \times 5^2 \times 7^2, \qquad x + y + z = 2r. \tag{1}$$

Since the bell-ringer could not work out the ages initially, there must be at least two possible sets of (x, y, z) satisfying (1) for a fixed r. After an exhaustive check, the only possibilities are $(5, 10, 49)$ and $(7, 7, 50)$, with $r = 32$.

The vicar being older than all three allows the bell-ringer to confirm the choice of $(x, y, z) = (5, 10, 49)$ with $v = 50$, since otherwise $v > 50$ and both sets are still possible.

Chocolate addiction

Willy has several jars filled with chocolates, none of which is empty. Each move he is allowed to either double the content of one jar, or eat one chocolate from every jar. Can Willy always empty all the jars using these moves?

Bonus: *If the doubling move is replaced by tripling, can Willy always empty all the jars?*

Solution by Joe Kupka: All Willy has to do is to eat one chocolate from each jar, until some jars have only a single chocolate left. Now double the content of these single chocolate jars and repeat the process. If originally the jar with the most chocolates had k chocolates, then after $k - 1$ eating moves, every jar has exactly one chocolate. Now simply eat the last chocolate in each jar to empty them all.

Bonus: It is not always possible to empty the jars if doubling is replaced by tripling. For example, consider the case where there are only two jars. Each move Willy either eats two chocolates, or adds $2c$ chocolates to one of the jars which had c chocolates previously. Note that both moves preserve the parity of the total number of chocolates. Hence if initially there was an odd number of chocolates in total, then it is impossible to empty both jars.

Odd polygons

Prove that it is never possible to tile a polygon with only odd integer side lengths using 1×2 *dominoes.*

Solution by Adrian Nelson: First note that if a polygon with integer side lengths can be tiled by unit squares, then it can be divided into unit squares using a standard grid. This can be proven inductively by removing one unit square at a time without disconnecting the polygon. Furthermore, if we colour the squares using a checkerboard pattern, each 2×1 domino covers one black and one white square. Hence any polygon tileable by dominoes covers an equal number of black and white squares.

Consider a polygon with only odd side lengths. We will walk around its perimeter, starting at a corner with, say, a black edge. As we proceed along a side, the edges alternate between black and white, finishing on a black edge. Then after a $90°$ or

$270°$ turn, we start the next side again at a black edge, and continue this process. So every side of the polygon has one more black edge than white. Hence in total, the perimeter contains S more black edges than white, where S is the number of sides.

Now every unit square has four edges, and every internal black edge is matched by an internal white edge. Hence there are $S/4 > 0$ more black than white squares, which is a contradiction. Therefore, it is never possible to tile a polygon with only odd integer side lengths using dominoes.

Probability problems

1. *On a circle, n points are chosen randomly. What is the probability that they all lie on a semicircular arc?*
2. *A strange machine takes a positive integer n as input and randomly outputs an integer between 1 and n inclusive. We start by giving the machine $n = 1000$, and continue to feed the output back into the machine as input. On average, how many times do we have to use the machine until it outputs the number 1?*

Solution by Ross Atkins:

1. Let the circumference of the circle be 1. The n points partition the circle into n arcs, whose lengths sum to 1. Choosing these points uniformly is equivalent to choosing $(x_1, \dots, x_n) \in \mathbb{R}^n$ uniformly from the set
$$S = \{(x_1, \dots, x_n) : \quad x_1 + \cdots + x_n = 1, \quad x_1, \dots, x_n \geq 0\}, \tag{2}$$
which is an $(n-1)$ dimensional simplex.
Having all points on a semicircle is equivalent to having $x_i \geq \frac{1}{2}$ for some i. Without the loss of generality, say $x_1 \geq \frac{1}{2}$ and write $y = x_1 - \frac{1}{2}$. Then (2) can be rewritten as
$$T = \{(y, x_2, \dots, x_n) : \quad y + x_2 + \cdots + x_n = \frac{1}{2}, \quad y, x_2, \dots, x_n \geq 0\}.$$
Now T is also an $(n-1)$ dimensional simplex, half as big as S in terms of side lengths. So T is $\frac{1}{2^{n-1}}$ as big as S in volume. Hence the probability of $x_1 \geq \frac{1}{2}$ is $\frac{1}{2^{n-1}}$.
Repeating the argument for $x_2 \geq \frac{1}{2}$ and so on, the required probability is therefore $\frac{n}{2^{n-1}}$.
2. Let a_n be the expected number of runs until the machine outputs 1, given that the machine has just output n. Then $a_1 = 0$ and
$$a_k = 1 + \frac{a_1 + a_2 + a_3 + \cdots + a_k}{k}, \qquad k \geq 2. \tag{3}$$
For $k = 2$ we quickly get $a_2 = 2$. Considering (3) for $k = n$ and $k = n + 1$, we get
$$(n+1)a_{n+1} - (n+1) = \sum_{i=1}^{n+1} a_i = a_{n+1} + \sum_{i=1}^{n} a_i = a_{n+1} + na_n - n.$$

Rearranging yields $a_{n+1} = a_n + \frac{1}{n}$ for all $n \geq 2$. Therefore for $n \geq 2$, the answer is one more than the harmonic series

$$a_n = 1 + \sum_{k=1}^{n-1} \frac{1}{k}.$$

For $n = 1000$ we have $a_n \approx 8.48$.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.