

Euler and the Mathematics Challenge for Young Australians

K.L. McAvaney*, P.J. Taylor** and S. Thornton***

Abstract

The Mathematics Challenge for Young Australians is an enrichment program for middle level school students. A graphical technique used in a solution to one of its recent problems provides a partial solution to Euler's celebrated 36 officers problem.

1. Introduction

The acclaimed eighteenth century Swiss mathematician Leonhard Euler (1707–1783) is firmly placed in the folklore of many branches of mathematics. Combinatorial design theory is one of them. In 1782 he proposed the seemingly simple thirty-six officers problem [6]:

Given 6 officer ranks and 6 regiments, is it possible to arrange 36 officers in a square of 6 rows and 6 columns so that each row and each column contains exactly one officer of each rank and exactly one officer from each regiment?

In modern parlance, Euler was asking for a pair of Latin squares of order six that are orthogonal, that is, no pair of corresponding entries occurs more than once. He found orthogonal pairs for all orders except twice the odd numbers. There are only two Latin squares of order 2 and they are not orthogonal. It wasn't until 1900 that order 6 was resolved. Gaston Tarry, a French public servant in Algeria, by systematically classifying and enumerating the thousands of Latin squares of order 6, showed that no two were orthogonal [7], [8]. This has since been confirmed by various algebraic and computational proofs. Euler's gap was finally closed in 1960 by R.C. Bose, S.S. Shrikhande, and E.T. Parker [2] who produced examples of orthogonal pairs of Latin squares for all other orders that are twice an odd number.

Attention subsequently turned to self-orthogonal Latin squares or SOLS. A Latin square is self-orthogonal if it is orthogonal not to itself, of course, but to its transpose. All main diagonal entries in a SOLS must be distinct and, by appropriate transpositions of rows and columns, we may assume it is idempotent, that is, the

*Australian Mathematics Trust, University of Canberra, ACT 2601.

Email: kevin.mcavaney@ozemail.com.au

**Australian Mathematics Trust, University of Canberra, ACT 2601.

Email: pjt013@gmail.com

***School of Education, Charles Darwin University, Casuarina, NT 0810.

Email: stephen.thornton@cdu.edu.au

diagonal entries are in their natural order. A quick check shows there are no SOLS of order 3. There are two idempotent SOLS of order 4, one the transpose of the other:

$$\begin{array}{cccc} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{array} \quad \begin{array}{cccc} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{array}$$

There are 12 idempotent SOLS of order 5 but, from the remarks above, no SOLS of order 6. At a conference in 1973, R.K. Brayton, D. Coppersmith and A.J. Hoffman showed that there are SOLS for all orders except 2, 3, and 6. The result was published in [3] and [4]. There is no known short elementary proof that no two Latin squares of order 6 are orthogonal, but is there such a proof that no SOLS of order 6 exists? We return to that question in Section 3.

2. A Mathematics Challenge problem

The Mathematics Challenge for Young Australians is an enrichment program for talented school students from Years 5 to 10. It is produced by the Australian Mathematics Trust and described on its website www.amt.canberra.edu.au/mcya.html. It consists of three stages: Challenge, Enrichment, and the Australian Intermediate Mathematics Olympiad. The Challenge stage started in 1992 and now has three sections: Primary for Years 5 and 6, Junior for Years 7 and 8, and Intermediate for Years 9 and 10. It attracts some 15 000 students mainly from Australasia and East Asia. It is held over a three-week period chosen by the school in the first half of the year. The Enrichment stage is held in the second half of the year and has six sections each based on a booklet of non-curriculum mathematics. The AIMO is the culminating stage consisting of a four-hour 10-question examination held in August. Performance in the MCYA is used to select students for the national and international mathematics olympiad training programs.

Motivated by Ian Anderson [1], the following problem was posed in the Intermediate section of the 2012 Challenge.

Tournaments with a Twist

The Bunalong Tennis Club is running a mixed doubles tournament for families from the district. Families enter one female and one male in the tournament. When the schedule is arranged, the players discover the twist: they never partner or play against their own family member.

The schedule is arranged so that:

- 1. each player plays against every person of the same gender exactly once*
- 2. each player plays against every person of the opposite gender, except for his or her family member, exactly once*
- 3. each player partners every person of the opposite gender, except for his or her family member, exactly once.*

Using the notation M_1 and F_1 for the Male and the Female in Family 1, M_2 and F_2 for Family 2 and so on, an example of an allowable match is $M_3F_1 \vee M_6F_4$.

- Explain why there cannot be fewer than four families.
- Give an example of such a schedule for four families.
- Give an example of such a schedule for five families.
- Find all such schedules for four families.

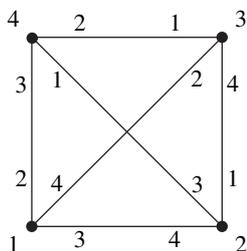
Anderson found examples of similar problems dating back to the late nineteenth century. He called them spouse-avoidance mixed double round robin (SAMDRR) tournaments. As explained by Brayton *et al.* [3], any SAMDRR tournament can be uniquely represented by an idempotent SOLS in the following way. For the match $M_iF_k \vee M_jF_l$ we insert k at the intersection of row i and column j and insert l at the intersection of row j and column i . For example, the SAMDRR tournament corresponding to the first SOLS of order 4 above is:

$$\begin{array}{ll} M_1F_3 \vee M_2F_4 & M_3F_1 \vee M_4F_2 \\ M_1F_4 \vee M_3F_2 & M_2F_3 \vee M_4F_1 \\ M_1F_2 \vee M_4F_3 & M_2F_1 \vee M_3F_4 \end{array}$$

The tournament corresponding to the second SOLS simply has the females in each of these matches reversed.

3. A solution from graph theory

To ease the task of assessing students' work, the Mathematics Challenge includes as many alternative solutions as possible in its teacher guide. Besides the natural table method of representing tournaments, a graphical method was suggested for Parts c and d in the problem above. An n family tournament is represented by a complete graph on n vertices (every pair of vertices is joined by an edge). The vertices correspond to the males and the edges correspond to their matches. Each edge has two labels, one at each end, corresponding to the females in the match. The female label that is closest to a male vertex is that male's partner in the match. For example, the graph for the 4-family tournament above is



These graphs are a convenient tool for constructing and analysing SAMDRR tournaments. It would be natural to call them tournament graphs but that name is used in graph theory for complete graphs in which every edge is directed. Instead,

somewhat reluctantly, we shall call them SAMDRR graphs. They have the following useful property.

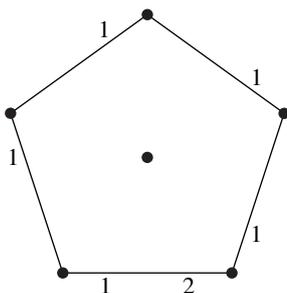
Lemma 3.1. The subgraph of an SAMDRR graph that consists of all the edges with a given label, k say, is a union of disjoint cycles.

Proof. M_k neither partners nor opposes F_k and each male except M_k partners and opposes F_k exactly once. Thus, at each vertex of the SAMDRR graph, except M_k , there is exactly one edge with label k close to that vertex. Hence, starting at any vertex except M_k , we may trace a path from vertex to vertex each time selecting the edge with label k close to that vertex. Since F_k does not oppose any male more than once, the path must terminate at the initial vertex thus forming a cycle. Repeating this procedure with any excluded males will produce a succession of disjoint cycles until all edges labeled k are exhausted. \square

We can now give a short elementary proof of the fact that there are no SOLS of order 6.

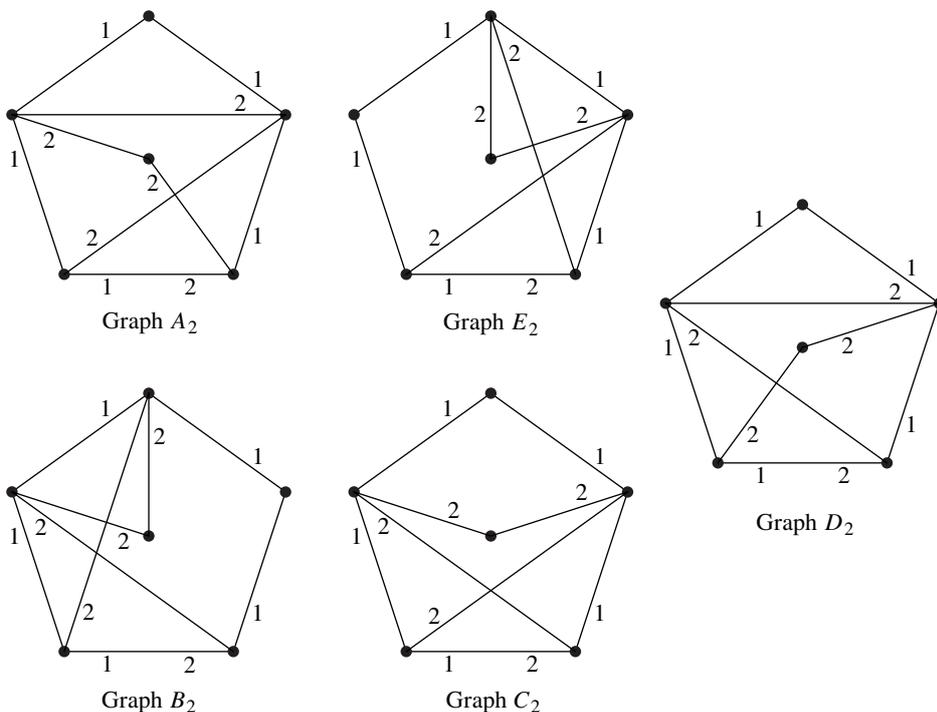
Theorem 3.1. *There is no SOLS of order 6.*

Proof. From the Lemma, all edges with label 1 in an SAMDRR graph of order 6 form a cycle of five edges. Since F_1 opposes every female exactly once, the four edges in the cycle must be labelled, in some order, 1 and 2, 1 and 3, 1 and 4, 1 and 5, and 1 and 6. Thus we have the following diagram with labels 3, 4, 5, 6 missing from the edges. The centre vertex is M_1 .



All the edges labelled 2 must also form a cycle of five edges. There are only five ways to fit such a cycle and avoid a second edge labelled 1 and 2.

In graph A_2 the bottom right edge with label 1 must also have label 3, 4, or 5. Suppose it has label 3. All the edges labelled 3 must also form a cycle of five edges. There are five ways to fit such a cycle and avoid a second edge labelled 1 and 3. These correspond to graphs A_2, B_2, C_2, D_2, E_2 with 2 replaced by 3, so we call them A_3, B_3, C_3, D_3, E_3 respectively. We fit each label 3 cycle to A_2 by rotating each of A_3, B_3, C_3, D_3, E_3 anticlockwise through 72° about the centre vertex and superimposing them in turn on A_2 . Perhaps the easiest way of doing this, and a lovely student exercise, is to photocopy these graphs onto an acetate sheet and rotate those copies on a copy of A_2 . In each case there is an edge label 2 and edge



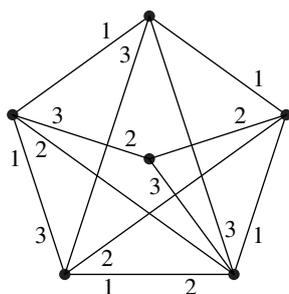
label 3 fighting for the same position or there are two edges labelled 2 and 3. So, in A_2 , there cannot be any other label on the bottom right edge with label 1.

Similarly the bottom right edge of E_2 with label 1 cannot carry any other label.

We may assume the edge on the bottom left of B_2 with label 1 has label 3. Superimposing A_3, B_3, C_3, D_3, E_3 in turn on B_2 again produces a contradiction.

Similarly the bottom left edge of D_2 with label 1 cannot carry any other label.

The bottom left edge of C_2 with label 1 may carry label 3, but only by superimposing D_3 :



Then the bottom right edge of this graph with label 1 cannot carry any other label.

This exhausts all possibilities so there is no SAMDRR graph of order 6, hence no SOLS of order 6. \square

While we were preparing this paper, Mike Newman drew our attention to the work by Burger *et al.* [5]. They use a different graphical method to show that there is no SOLS of order 6. Instead of the SAMDRR tournament representation of a SOLS, they use a direct representation of a SOLS based on its transversals. For a SOLS of order n it is a directed graph of order $n(n-1)/2$ in which the arcs are partitioned into n directed cycles of length $n-1$. A fairly lengthy argument shows that their graph of any SOLS of order 6 has minimum directed cycle length 5 and is consequently impossible to construct.

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