Enhancing the Jordan canonical form

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Abstract

The Jordan canonical form parametrises similarity classes in the nilpotent cone $\mathcal{N}_n$, consisting of $n \times n$ nilpotent complex matrices, by partitions of n. Achar and Henderson (2008) extended this and other well-known results about $\mathcal{N}_n$ to the case of the *enhanced nilpotent cone* $\mathbb{C}^n \times \mathcal{N}_n$.

1. Jordan canonical form

The Jordan canonical form (JCF), introduced in 1870 [10], is one of the most useful tools in linear algebra. As an illustration, consider the following result. (Here and throughout, all matrices have entries in $\mathbb{C}$.)

Proposition 1.1. *Let A be an invertible $n \times n$ matrix. Then A has a square root: that is, there exists an $n \times n$ matrix B such that $B^2 = A$.*

The $n = 1$ case of Proposition 1.1 just says that any nonzero complex number a has a square root $\sqrt{a}$, which is one of the well-known advantages of working in $\mathbb{C}$. The $n = 2$ case is already a bit tricky, if we use only the definition of matrix multiplication: it amounts to showing that there is a solution to a system of four degree-2 equations in the four unknown entries of B. (This is not automatic, as shown by the existence of noninvertible 2×2 matrices with no square root.) Attempting to prove the $n = 3$ case this way would be foolish.

A better approach to Proposition 1.1 is the maxim 'use the symmetry of the problem to reduce to a special case'. Remember that $n \times n$ matrices A and A' are said to be *similar* if $A' = XAX^{-1}$ for some invertible matrix X. If this is the case, then A has a square root if and only if A' has a square root, because of the easy observation that $(XBX^{-1})^2 = XB^2X^{-1}$. This means that we only need to consider one representative A from each similarity class of $n \times n$ invertible matrices. The JCF theorem gives us such a representative.

Theorem 1.1 (Jordan canonical form)**.** *Every similarity class of $n \times n$ matrices contains a matrix A that is block-diagonal with diagonal blocks $J_{\ell_1}(a_1), J_{\ell_2}(a_2), \ldots,$*

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$J_{\ell_k}(a_k)$ for some $\ell_i \in \mathbb{Z}^+$ and $a_i \in \mathbb{C}$, where

$$J_\ell(a) = \begin{pmatrix} a & 1 & 0 & \cdots & 0 & 0 \\ 0 & a & 1 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 1 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix} \qquad (\ell \text{ rows and columns}).$$

Moreover, A is unique except that the blocks $J_{\ell_i}(a_i)$ can be put in a different order.

The proof of Proposition 1.1 is now straightforward. It suffices to show that A has a square root when A is in Jordan canonical form as in Theorem 1.1; the assumption that A is invertible means that all a_i are nonzero. Since A is block-diagonal, it suffices to show that each block $J_\ell(a)$ with $a \neq 0$ has a square root. The naive guess that $J_\ell(\sqrt{a})^2 = J_\ell(a)$ doesn't work, but it is easy to check that $J_\ell(\sqrt{a})^2$ and $J_\ell(a)$ are similar when $a \neq 0$, which is enough to finish the proof.

There are obviously more questions along the same lines. Which noninvertible $n \times n$ matrices have a square root? How about a kth root? See [5] for the answers in terms of the JCF. The main point here is the great advantage of being able to reduce to such special matrices.

2. The nilpotent cone

If we examine the proof of Theorem 1.1 in an undergraduate textbook such as [8], we see that it is really two theorems rolled into one. First there is the theorem that if A is an $n \times n$ matrix, then $\mathbb{C}^n$ is the direct sum of the generalised eigenspaces of A. Having proved that, we can restrict attention to matrices with a single eigenvalue, and it doesn't really matter what that eigenvalue is. So the second theorem is about similarity classes of *nilpotent* matrices, those whose sole eigenvalue is zero. In this context it is sensible to specify an ordering on the Jordan blocks, giving a tighter statement.

Theorem 2.1 (Jordan canonical form, nilpotent case)**.** *The similarity classes of $n \times n$ nilpotent matrices are in bijection with the weakly decreasing sequences $(\ell_1, \ldots, \ell_k)$ of positive integers whose sum is n. Given $(\ell_1, \ldots, \ell_k)$, the corresponding similarity class $\mathcal{O}_{(\ell_1,\ldots,\ell_k)}$ is the one containing the matrix that is block-diagonal with diagonal blocks $J_{\ell_1}(0), \ldots, J_{\ell_k}(0)$.*

These weakly decreasing sequences $(\ell_1, \ldots, \ell_k)$ are known as the *partitions* of n, and arise in many different parts of mathematics [3]. It is conventional to represent such a partition by its *Ferrers diagram*, a left-justified array of boxes where there are ℓ_1 boxes in the first row, ℓ_2 in the second row, and so on. For example,

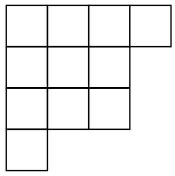

Figure 1.

represents the partition $(4, 3, 3, 1)$ of 11. The Ferrers diagram of $(\ell_1, \ldots, \ell_k)$ shows how a matrix A in the similarity class $\mathcal{O}_{(\ell_1,\ldots,\ell_k)}$ acts on the space of column vectors $\mathbb{C}^n$. Each box represents a basis element of $\mathbb{C}^n$; this could be the usual basis if A is in Jordan canonical form, but otherwise is a basis adapted to A. When we multiply A by a box, we get the box to the left of it in the same row; or, if we multiply A by a box that is already at the left-hand end of its row, we get zero. The 0-eigenspace of A is thus spanned by the boxes in the first column. For example, if $A \in \mathcal{O}_{(4,3,3,1)}$, we can read off from Figure (1) that the 0-eigenspace of A is 4-dimensional, so the rank of A is $11 - 4 = 7$. Similarly, the rank of A^2 is 4, the rank of A^3 is 1, and A^4 is the zero matrix.

The set $\mathcal{N}_n$ of all $n \times n$ nilpotent matrices is called the *nilpotent cone*. When $n = 2$, this is the set of matrices $\left(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right)$ where $x^2 + yz = 0$ (if x, y, z were real numbers, this would be the equation of a cone in the sense of conic sections). Theorem 2.1 tells us that $\mathcal{N}_n$ is a disjoint union of finitely many similarity classes $\mathcal{O}_{(\ell_1,\ldots,\ell_k)}$, each of which individually is homogeneous, meaning that any two points are equivalent; but $\mathcal{N}_n$ as a whole has singularities, for example, the vertex of the cone in the $n = 2$ case. The geometry of how the classes $\mathcal{O}_{(\ell_1,\ldots,\ell_k)}$ fit together has many surprising connections to representation theory and combinatorics.

A fundamental question here is which similarity classes are limiting cases of which other ones — in other words, when does $\mathcal{O}_{(\ell_1,\ldots,\ell_k)}$ belong to the topological closure of $\mathcal{O}_{(\ell'_1,\ldots,\ell'_{k'})}$? For example, the fact that

$$\begin{pmatrix} 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{belongs to} \quad \begin{cases} \mathcal{O}_{(3,1)} & \text{if } x \neq 0 \\ \mathcal{O}_{(2,2)} & \text{if } x = 0 \end{cases} \tag{2.1}$$

shows that $\mathcal{O}_{(2,2)} \subset \overline{\mathcal{O}_{(3,1)}}$. In the converse direction, if B is the limit of a sequence of matrices in $\mathcal{O}_{(4,3,3,1)}$, then from the calculations above we can deduce that the rank of B is at most 7, the rank of B^2 is at most 4, and so on, which puts restrictions on which classes can appear in $\overline{\mathcal{O}_{(4,3,3,1)}}$. The complete answer has been known for over fifty years [7]:

Theorem 2.2. *If $(\ell_1, \ldots, \ell_k)$ and $(\ell'_1, \ldots, \ell'_{k'})$ are partitions of n, then $\mathcal{O}_{(\ell_1,\ldots,\ell_k)} \subset \overline{\mathcal{O}_{(\ell'_1,\ldots,\ell'_{k'})}}$ if and only if*

$$\ell_1 + \ell_2 + \cdots + \ell_m \leq \ell'_1 + \ell'_2 + \cdots + \ell'_m \quad \text{for all } m \geq 1.$$

Here we use the convention that $\ell_i = 0$ if $i > k$ and $\ell'_i = 0$ if $i > k'$.

The collection of inequalities in Theorem 2.2 is known as the *dominance* relation on partitions, and also appears in the combinatorics of symmetric functions and the representation theory of the symmetric group [15]. The deeper connection behind this apparent coincidence was revealed by Lusztig [12]. He studied *perverse sheaves* on the nilpotent cone, which are topological objects encoding the singularities of the closures $\overline{\mathcal{O}_{(\ell_1,\ldots,\ell_k)}}$. He showed that these sheaves were related to the irreducible representations of the symmetric group via the *Springer correspondence*, and that the dimensions of their stalks equal the coefficients of the combinatorial *Kostka polynomials.*

The article [12] was the start of Lusztig's work on *character sheaves* in the 1980s, which culminated in the solution of one of the major open problems in algebra: computing the character tables of finite simple groups of Lie type [13]. Perverse sheaves on the nilpotent cone have become a prototypical example for the field of geometric representation theory, which features similar constructions on many other spaces [14]. And there is still more to say about $\mathcal{N}_n$ itself: it plays a key role in the recent work of Bezrukavnikov and co-authors on modular representation theory, such as in [4].

3. The enhanced nilpotent cone

Theorem 1.1 is fine for answering questions about a single matrix, but what about questions involving both a matrix and a vector, such as:

Question 3.1. *Let A be an $n \times n$ matrix and v an eigenvector of A. When does A have a square root B for which v is also an eigenvector?*

The JCF theorem on its own is not enough for Question 3.1: it needs to be 'enhanced' to describe similarity classes of pairs (v, A), where v is a vector in $\mathbb{C}^n$ and A is an $n \times n$ matrix. Here we consider two pairs (v, A) and (v', A') to be similar if there is an invertible matrix X such that $v' = Xv$ and $A' = XAX^{-1}$. This is the equivalence relation that is most natural in the sense that it respects properties such as the eigenvector equation $Av = \lambda v$, or, say, the property that v belongs to the image of A^3.

It is again easy to reduce to the case where A is nilpotent. So we need only consider the similarity classes of pairs (v, A) that belong to the *enhanced nilpotent cone* $\mathbb{C}^n \times \mathcal{N}_n$. Within any similarity class, there is clearly a pair (v, A) for which A is in Jordan canonical form. But even after fixing A to be in this nice form, we still have the freedom to manipulate v. Indeed, we can multiply v by any invertible matrix X such that $XAX^{-1} = A$.

Recall that the boxes of the Ferrers diagram represent a basis adapted to A. Specifying v is the same as specifying the coefficients of the basis vectors in v, which we can do pictorially by writing the coefficients in the boxes. An operation of multiplying v by X such that $XAX^{-1} = A$ can be broken into steps that are analogous to the elementary row operations on matrices:

$$\begin{array}{cccc}5&1&0&0\\2&0&0&\\0&3&0&\\0&&&\end{array} \xrightarrow{\text{scale } R_3} \begin{array}{cccc}5&1&0&0\\2&0&0&\\0&1&0&\\0&&&\end{array} \xrightarrow{R_2\to R_2+R_3} \begin{array}{cccc}5&1&0&0\\2&1&0&\\0&1&0&\\0&&&\end{array}$$
$$\xrightarrow{R_2\to R_2-2R_1^{\text{segment}}} \begin{array}{cccc}5&1&0&0\\0&1&0&\\0&1&0&\\0&&&\end{array} \xrightarrow{R_1\to R_1-5R_1^{\text{segment}}} \begin{array}{cccc}0&1&0&0\\0&1&0&\\0&1&0&\\0&&&\end{array} \tag{3.1}$$

Here, the last two steps illustrate an operation that is not familiar from first-year linear algebra, namely adding to one row of the diagram a multiple of a right-hand segment of another row (or even the same row), padded out with zeroes if it is not long enough (the right-hand segment used above is shown with a black border).

The upshot is the following result, proved independently by Achar and Henderson [1] and by Travkin [17].

Theorem 3.1. *The similarity classes in $\mathbb{C}^n \times \mathcal{N}_n$ are in bijection with the bipartitions of n: that is, ordered pairs $((\ell_1, \ell_2, \dots), (r_1, r_2, \dots))$ where $(\ell_1, \ell_2, \dots)$ is a partition of m and $(r_1, r_2, \dots)$ is a partition of $n - m$ for some $0 \le m \le n$. Given such a bipartition, the corresponding class $\mathcal{O}_{(\ell_i),(r_i)}$ is the one containing (v, A) where A is the JCF matrix in $\mathcal{O}_{\ell_1+r_1,\ell_2+r_2,\dots}$ and v is the sum of the basis vectors corresponding to the ℓ_ith box in the ith row as i varies.*

For example, the final array in (3.1) depicts the standard representative in the class $\mathcal{O}_{(2,2,2),(2,1,1,1)}$. With Theorem 3.1 in hand, answering Question 3.1 and others like it is an easy matter.

Aside from linear algebra applications, Theorem 3.1 allows us to study the enhanced nilpotent cone $\mathbb{C}^n \times \mathcal{N}_n$ along the lines of the rich theory of $\mathcal{N}_n$. In [1] we proved an analogue of Theorem 2.2:

Theorem 3.2. *If $((\ell_1, \ell_2, \dots), (r_1, r_2, \dots))$ and $((\ell'_1, \ell'_2, \dots), (r'_1, r'_2, \dots))$ are bipartitions of n, then $\mathcal{O}_{(\ell_i),(r_i)} \subset \overline{\mathcal{O}_{(\ell'_i),(r'_i)}}$ if and only if*

$$\ell_1 + r_1 + \ell_2 + r_2 + \cdots + \ell_m \le \ell'_1 + r'_1 + \ell'_2 + r'_2 \cdots + \ell'_m$$

and

$$\ell_1 + r_1 + \ell_2 + r_2 + \cdots + \ell_m + r_m \le \ell'_1 + r'_1 + \ell'_2 + r'_2 \cdots + \ell'_m + r'_m$$

for all $m \ge 1$.

We also generalised Lusztig's result from [12], showing that the dimensions of stalks of perverse sheaves on $\mathbb{C}^n \times \mathcal{N}_n$ equal the coefficients of some variants of Kostka polynomials defined by Shoji [16]. These coefficients were not previously known to be nonnegative.

The project of finding enhanced analogues of results about the nilpotent cone has been carried further in [2], [6], [9]. By the work of Kato [11], perverse sheaves on the enhanced nilpotent cone, and on a variant he defined called the *exotic nilpotent cone*, are related to representations of p-adic groups, so they seem likely to remain important.

References

[1] Achar, P.N. and Henderson, A. (2008). Orbit closures in the enhanced nilpotent cone. *Adv. Math.* **219,** 27–62.

[2] Achar, P.N., Henderson, A. and Jones, B.F. (2011). Normality of orbit closures in the enhanced nilpotent cone. *Nagoya Math. J.* **203,** 1–45.

[3] Andrews, G. E. (1998). *The Theory of Partitions.* Cambridge University Press, Cambridge.

[4] Bezrukavnikov, R., Mirković, I. and Rumynin, D. (2008). Localization of modules for a semisimple Lie algebra in prime characteristic (with an Appendix by R. Bezrukavnikov and S. Riche: Computation for sl(3)). *Ann. Math.* **167,** 945–991.

[5] Borwein, J.M. and Richmond, B. (1984). How many matrices have roots? *Canad. J. Math.* **36,** 286–299.

[6] Finkelberg, M., Ginzburg, V. and Travkin, R. (2009). Mirabolic affine Grassmannian and character sheaves. *Sel. Math. New Ser.* **14**, no.3–4, 607–628.

[7] Gerstenhaber, M. (1959). On nilalgebras and linear varieties of nilpotent matrices, III. *Ann. Math.* (Second Series) **70,** 167–205.

[8] Herstein, I.N. (1975) *Topics in Algebra*, 2nd edn. Xerox College Publishing, Lexington, MA.

[9] Johnson, C.P. Enhanced nilpotent representations of a cyclic quiver. arXiv:1004.3595v1.

[10] Jordan, C. (1989). Traité des substitutions et des équations algébriques. reprint of the 1870 original, Les Grands Classiques Gauthier–Villars, Éditions Jacques Gabay, Sceaux.

[11] Kato, S. (2009). An exotic Deligne-Langlands correspondence for symplectic groups. *Duke Math. J.* **148,** 305–371.

[12] Lusztig, G. (1981). Green polynomials and singularities of unipotent classes. *Adv. Math.* **42,** 169–178.

[13] Lusztig, G. (1984) Characters of reductive groups over finite fields. In: *Proc. Internat. Congr. Math.* (Warsaw, 1983), Vol. 1, pp. 877–880. PWN, Warsaw.

[14] Lusztig, G. (1991). Intersection cohomology methods in representation theory. In: *Proc. Internat. Congr. Math.* (Kyoto, 1990), Vols. I, II, pp. 155–174. Math. Soc. Japan, Tokyo.

[15] Sagan, B. (2001). *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions* (Graduate Texts Math. **203**), 2nd edn. Springer, New York.

[16] Shoji, T. (2004). Green functions attached to limit symbols. In: *Representation Theory of Algebraic Groups and Quantum Groups* (Adv. Stud. Pure Math. **40**), pp. 443–467. Math. Soc. Japan, Tokyo.

[17] Travkin, R. (2009). Mirabolic Robinson–Schensted–Knuth correspondence. *Selecta Math. New Ser.* **14,** 727–758.