Configuration spaces in topology and geometry

Craig Westerland∗

The $n$th configuration space, $\text{Conf}_n(X)$, of a topological space $X$, is the space of $n$ distinct points in $X$. In formulas,

$$\text{Conf}_n(X) := \{ (x_1, \ldots, x_n) \mid x_i \neq x_j \text{ if } i \neq j \}.$$ 

This is often called the ordered configuration space. There is a natural action of the symmetric group $S_n$ on $\text{Conf}_n(X)$ which reorders the indices of the $n$-tuple; the quotient $\text{Conf}_n(X)/S_n$ by this action is therefore the unordered configuration space.

This family of spaces has been studied from many points of view. For instance, in gravitation, $\text{Conf}_n(\mathbb{R}^3)$ is the natural home for the $n$ body problem (see [6] for the interesting history of this topic). Configurations of points in $\mathbb{R}^3$ that are required to conform to a given geometry (say, that of a robot’s arm) are employed in robotics and motion planning; see, for example, [10]. In this note, we will discuss the appearance of these spaces in homotopy theory and algebraic geometry.

**Example 1.** Let $I$ denote the open interval $I = (0, 1)$, and examine $\text{Conf}_n(I)/S_n$. Up to reordering, $n$ distinct points in $I$ are given by an increasing sequence $0 < t_1 < t_2 < \cdots < t_n < 1$ of real numbers. The collection of such $n$-tuples $(t_1, \ldots, t_n)$ is called the open $n$-dimensional simplex. For instance, when $n = 1, 2, 3$, these are easily seen to be $I$, an open triangle, and an open tetrahedron.

So $\text{Conf}_n(I)/S_n$ is an open simplex; $\text{Conf}_n(I)$ is simply a disjoint union of $n!$ copies of this, since there are precisely $n!$ different reorderings of the $t_i$.

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∗Department of Mathematics and Statistics, University of Melbourne, Parkville, VIC 3010.
Email: C.Westerland@ms.unimelb.edu.au

†Conrad Shawcross’s kinetic sculpture ‘Loop System Quintet’, on display at MONA in Hobart, Tasmania, gives a wonderful realisation of such a configuration space.
Configuration spaces for different spaces \( X \) and \( X' \) are often related if \( X' \) is obtained from \( X \) by removing a finite set of points.

**Example 2.** Let \( S^1 \) denote the unit circle in \( \mathbb{C} \) and \( I = (0, 1) \). There is a homeomorphism \( f: S^1 \times \text{Conf}_n(I) \to \text{Conf}_{n+1}(S^1) \) given by

\[
f(z, t_1, \ldots, t_n) = (z, ze^{2\pi it_1}, \ldots, ze^{2\pi it_n}).
\]

Thus \( \text{Conf}_{n+1}(S^1) \) is simply the product of a circle with a union of \( n! \) open simplices. The action of \( S_{n+1} \) is somewhat harder to visualise in this description.

We are interested in the topology of configuration spaces; at first blush, let us examine the fundamental group \( \pi_1(\text{Conf}_n(X)) \).

**Example 3.** First, note that a path in \( \text{Conf}_n(X) \) consists of an \( n \)-tuple of paths of distinct points in \( X \); this is a loop if it starts and ends at the same configuration. Replacing such a family of paths with their graphs, we see that a loop in \( \text{Conf}_n(X) \) can be regarded as an \( n \)-tuple of nonintersecting arcs in \( X \times I \) that begin and end at the same collection of \( n \) points. This is called an \( n \)-strand braid in \( X \times I \); when \( X = \mathbb{R}^2 \) this recovers the usual notion of braids in 3-space, and indeed the fundamental group \( \pi_1(\text{Conf}_n(\mathbb{R}^2)) =: \text{P} \beta_n \) is Artin’s (pure) braid group \([1]\).

In fact, to the eyes of homotopy theory, this is a complete description of \( \text{Conf}_n(\mathbb{R}^2) \). Through an inductive procedure similar to that of Example 2, one can show that all of the higher homotopy groups of \( \text{Conf}_n(\mathbb{R}^2) \) vanish \([9]\), and so all homotopy theoretic questions about \( \text{Conf}_n(\mathbb{R}^2) \) may be reduced to algebraic questions about the braid group.

**Function spaces.** One application of configuration spaces is to the study of spaces of functions. Here, if \( X \) and \( Y \) are topological spaces, we will write \( \text{Map}(X, Y) \) for the topological space of all continuous functions from \( X \) to \( Y \), equipped with the compact-open topology. If \( X \) and \( Y \) have basepoints, we will write \( \text{Map}^*_n(X, Y) \) for the subspace of maps that carry the basepoint of \( X \) to that of \( Y \). These are very large (usually infinite dimensional) spaces.

Finite dimensional approximations to \( \text{Map}(X, Y) \) can sometimes be given using configuration spaces. For instance, there is an ‘electrostatic map’ \([11]\)

\[
e: \text{Conf}_n(\mathbb{R}^k)/S_n \to \text{Map}^*_n(S^k, S^k).
\]

(Here \( \text{Map}_n \) indicates that the degree of these maps is \( n \).) One may assign an electric charge to each element of a configuration \( \mathbf{x} := (x_1, \ldots, x_n) \) in \( \mathbb{R}^k \). The associated electric field (with poles at the \( x_i \)) may be regarded as a function \( e(\mathbf{x}) \) from \( \mathbb{R}^k \) to \( S^k = \mathbb{R}^k \cup \{\infty\} \) which extends naturally over infinity. It is a consequence of the work of many authors (for example, \([13]\), \([2]\), \([11]\), \([14]\)) that configuration spaces ‘see’ large parts of the topology of function spaces, for example:
Theorem 1. The induced map \( e_* : H_p(\text{Conf}_n(\mathbb{R}^k)/S_n) \to H_p(\text{Map}_n^*(S^k, S^k)) \) is an isomorphism in homology in dimensions \( p \leq n/2 \).

When \( k = 1 \), we note that by Example 1, the domain of \( e \) is a simplex, and thus contractible. The codomain \( \text{Map}_n^*(S^1, S^1) \) is also contractible to a standard degree \( n \) function \( g_n(z) = z^n \). That is, every degree \( n \) map \( f : S^1 \to S^1 \) may be lifted over \( \exp : \mathbb{R} \to S^1 \) to a map \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) (carrying \( \mathbb{Z} \) to \( n\mathbb{Z} \)), where it may be deformed linearly to \( \tilde{g}_n(x) = nx \). Theorem 1 is, in this case, a triviality.

In contrast, in dimension 2, Theorem 1 suggests that the homotopy theory of \( S^2 \) — that is, the study of the (rather mysterious) groups \( \pi_2 + m(S^2) \cong \pi_m(\text{Map}_n^*(S^2, S^2)) \) — may be approached using braid groups. This idea has been realised in the remarkable work of Fred Cohen and Jie Wu [5], who relate those homotopy groups to the descending central series in \( P\beta_n \).

Moduli spaces. Configuration spaces are also closely related to moduli spaces. The moduli space of \( n \) points on the Riemann sphere\(^2\) is the set of \( n \)-tuples in \( S^2 \), up to conformal automorphisms of \( S^2 \), that is:

\[
M_{0,n} := \text{Conf}_n(S^2)/\text{Aut}(S^2).
\]

Here \( \text{Aut}(S^2) = \text{PSL}_2(\mathbb{C}) \) acts on \( S^2 = \mathbb{C} \cup \{\infty\} \) (and hence \( n \)-tuples in \( S^2 \)) through Möbius transformations. Noting that for any \( x_1 \in S^2 \), there is a Möbius transformation \( T \) carrying \( x_1 \) to \( \infty \), we have a homeomorphism

\[
M_{0,n} \cong \text{Conf}_{n-1}(\mathbb{C})/\text{Aut}(\mathbb{C}) = \text{Conf}_{n-1}(\mathbb{C})/\text{Aff}(\mathbb{C})
\]

obtained by applying \( T \) to a configuration, and dropping \( \infty \) from the configuration. Lastly, the group \( \text{Aff}(\mathbb{C}) = \mathbb{C}^\ast \ltimes \mathbb{C} \) acts by affine transformations on the plane; we note that it is homotopy equivalent to the subgroup \( S^1 \) of rotations.

Combining this with the approximation from the electrostatic map, we see that the homology of \( M_{0,n}/S_n \) is isomorphic in a range to that of \( \text{Map}_{n-1}(S^2, S^2)/S^1 \), where the circle group acts on functions by rotating the codomain. That homology is quite complex indeed (see, for example, [4]), but is in fact entirely torsion\(^3\)! This gives the surprising result:

\[
H_p(M_{0,n}/S_n; \mathbb{Q}) = 0, \quad \text{if} \ p > 0. \tag{1.1}
\]

Hurwitz spaces. We would like to use these sorts of methods to compute algebro-topological invariants of other families of moduli spaces. Configuration space techniques are particularly well adapted to moduli spaces of surfaces equipped with structures that degenerate at a finite set of points. A good example are Hurwitz spaces — moduli spaces of branched covers of Riemann surfaces.

Let \( G \) be a finite group and \( c \subseteq G \) a union of conjugacy classes. A \((G, c)\)-branched cover of the sphere is a Riemann surface \( \Sigma \) equipped with a map \( p : \Sigma \to S^2 \) which, away from a set of \( n \) points in \( S^2 \), is an analytic, regular covering space with Galois group \( G \). Furthermore, we insist that the monodromy of the cover around the

\(^2\)For surfaces of positive genus, the moduli space is slightly more complicated, and involves the space of constant curvature metrics on the surface.

\(^3\)This is related to the fact that \( \pi_k(S^3) \) is torsion except when \( k = 2 \) or 3.
branch points lie in \( c \). The moduli space of such maps \( p \) will be denoted \( \text{Hur}^c_{G,n} \); this is the set of all such maps up to conformal automorphisms of \( S^2 \).

The topology of \( \text{Hur}^c_{G,n} \) is not difficult to describe: the forgetful map \( \Phi : \text{Hur}^c_{G,n} \to \mathcal{M}_{0,n}/S_n \) which carries \( p \) to its branch locus is a covering space. Now, a branched cover of \( S^2 \) may be reconstructed from its monodromy around the branch points. Thus the fibre of \( \Phi \) over a point \( [x_1, \ldots, x_n] \in \mathcal{M}_{0,n}/S_n \) is the set of possible values for that monodromy. This is the set

\[
S = \{ (g_1, \ldots, g_n) \in c^{\times n} | g_1 \cdots g_n = 1 \}.
\]

(The product of the local monodromies must be 1 in order for the cover to extend from a neighborhood of the branch locus over the rest of the sphere.) An explicit formula for the action of the (spherical) braid group \( \pi_1(\mathcal{M}_{0,n}/S_n) = \beta_n/Z(\beta_n) \) on \( S \) associated to this cover is given by

\[
\sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n),
\]

where \( \sigma_i \) is the braid that swaps the \( i \)th and \( i + 1 \)st strands.

While this is a beautiful description of \( \text{Hur}^c_{G,n} \), it is not one that immediately lends itself to computations of homology groups. In joint work with Jordan Ellenberg and Akshay Venkatesh [7, 8], we have adapted the classical techniques to this setting, and computed the homology of \( \text{Hur}^c_{G,n} \) for large values of \( n \). As in Theorem 1, it is given in terms of a space \( \text{Map}^*_n(S^2, A(G, c)) \) of functions from \( S^2 \) (that is, the base of the branched covering) into a certain classifying space \( A(G, c) \) for branched covers. The topology of the space \( A(G, c) \) is quite complicated. However, to the eyes of rational homology, it is nearly indistinguishable from \( S^2 \). A sample result, along the lines of equation (1.1), is:

**Theorem 2.** Let \( G \) be a group of order \( 2^p \), with \( p \) odd; let \( c \) be the conjugacy class of involutions and \( A \) the unique normal subgroup of \( G \) of order \( p \). Then there exists a constant\(^4 \alpha > 0 \) so that each component of \( \text{Hur}^c_{G,n} \) has vanishing positive Betti numbers in dimensions less than \( \alpha n \).

Using techniques of \( \acute{e} \)tale cohomology, this has, as a corollary, a function field analog of the Cohen-Lenstra heuristics [3] on the distribution of class groups of imaginary quadratic number fields. For function fields, the corresponding heuristics concern the statistics of hyperelliptic curves (degree 2 ramified covers of \( \mathbb{P}^1 \)) over \( \mathbb{F}_q \) with prescribed class group (or Picard group). Now, the Hurwitz space in question parameterises branched \( G \) covers of \( \mathbb{P}^1 \) with ramification occurring away from \( A < G \); if \( A \) is abelian, this is the same as a hyperelliptic curve \( C \) equipped with a surjection \( \text{Pic}(C) \to A \). Thus \( \text{Hur}^c_{G,n} \) is the home of the counting problem that the Cohen-Lenstra heuristics present; understanding its cohomology leads to a proof of the heuristics.

\(^4\)Here \( \alpha \) depends upon \( G \), and is much smaller than the constant \( \frac{1}{2} \) of Theorem 1.
References


Craig Westerland received his PhD from the University of Michigan in 2004 under the supervision of Igor Kriz. He spent the next four years at postdoctoral positions at the Institute for Advanced Study in Princeton, the University of Wisconsin, the University of Copenhagen, and the Mathematical Sciences Research Institute in Berkeley. He joined the mathematics department at the University of Melbourne in 2008 as lecturer, and currently is an ARC Future Fellow. His research interests centre around algebraic topology and its applications to number theory, geometry, and mathematical physics.