Puzzle corner

Ivan Guo[1]

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner No. 21. Each issue will include a handful of fun, yet intriguing, puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes SA 5095.

The deadline for submission of solutions for Puzzle Corner 21 is 1 May 2011. The solutions to Puzzle Corner 21 will appear in Puzzle Corner 23 in the July 2011 issue of the *Gazette*.

Combo cube

Eight cubes are marked with one dot on two opposite faces, two dots on two opposite faces and three dots on two opposite faces. The eight cubes are put together to form a $2 \times 2 \times 2$ cube. The dots on each face of the large cube are counted, to get six totals. Can the cubes be arranged so that the six totals are consecutive integers?

Curious cut

Is it possible to cut the shape on the left into two pieces, and join them together again to form the shape on the right?

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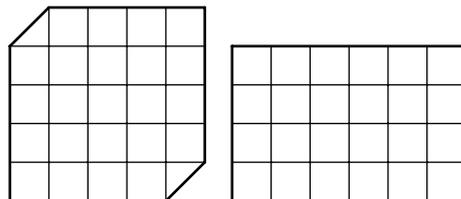
[1]E-mail: ivanguo1986@gmail.com

Stretchy string

A rubber string is 10 metres long. There is a worm crawling from one end toward the other at a speed of 1 metre per hour. Upon the passing of each hour, the rubber string is stretched uniformly to become 1 metre longer than it just was. Will the worm ever reach the other end of the string?

Genius Loci

Imagine yourself lost in a desert, searching for the village of Kyklos. Just when you think you will never meet another human being, a robed figure appears. Lost and exhausted, you ask for directions.

'Ahh, Kyklos! I was there this morning. Sorry I cannot tell you where it is or lead you there. They do not welcome outsiders.'

'Oh for crying out loud, I've been searching for it for weeks! You must help me. I'm running low on supplies, you are my last hope!' you beg.

'Fine, I'll give you a clue. The path I took to get here is in the shape of a perfect semicircular arc. And also, on my way here I visited the Shrine of the Serpents.' The figure gestures towards the setting sun. 'The Shrine is 10 miles that way.'

'But, the village could still be any ...'. Before you can finish, a sudden gust of wind sweeps through the area, knocking you off your feet. When the sand settles, the mysterious figure has vanished, leaving you with nothing but sheer frustration.

Even though there is no way to pinpoint the village of Kyklos exactly from this mumbo jumbo, could you at least narrow down its possible locations?

Poisoned prisoners

The king has 500 barrels of wine, but one of them is poisoned. Anyone drinking the poisoned wine will die within 12 hours. The king has four prisoners whom he is willing to sacrifice in order to find the poisoned barrel. Can this be done within 48 hours?

Twice or thrice

1. Show that there exists a unique function $f : \mathbb{N}^+ \to \mathbb{N}^+$ which is increasing and satisfies the equation

$$f(f(n)) = 3n$$

   for all $n \in \mathbb{N}^+$.

2. Show that there exist at least two functions $f : \mathbb{R} \to \mathbb{R}$ which are increasing and satisfy the equation

$$f(f(x)) = 3x$$

   for all $x \in \mathbb{R}$.
   Bonus: Can you find more of these functions?

Solutions to Puzzle Corner 19

The \$50 book voucher for the best submission to Puzzle Corner 19 is awarded to Alan Jones. Congratulations!

Tennis tournament

Solution by Leon Poladian: The second-best player can only ever lose to the best player. The best player has participated in eight matches and defeated eight opponents. So the second best player is amongst these eight opponents. It will thus take a further seven games to identify the second best player.

Paper route

Solution by David Angell: Represent the street by a row of 40 black dots (those houses that will receive newspapers) and 10 white dots (those that won't). We have to count the number of arrangements in which at least two black dots separate each pair of consecutive white dots.

As there are nine gaps between the white dots, we will set aside 18 black dots for the moment. Start with an arbitrary arrangement of the remaining 22 black dots and 10 white dots. We can insert the 18 reserved black dots into the nine gaps to ensure that each gap has at least two black dots, thus creating an arrangement satisfying the original conditions. In fact this is a bijection because the move is reversible from any valid original arrangement of 40 dots.

Therefore the required answer is also the number of ways to arrange 22 black dots and 10 white dots in a row, which is

$$\binom{32}{10} = 64512240.$$

Guessing game

Solution by James East: Let the apprentices be A, B and C, in order of speaking, with numbers a, b and c respectively. Since the numbers are non-negative, we must have $a, b, c \leq 12$. If a was even, then A could not have known that $b \neq c$, so a must be odd. Similarly, b is also odd. Furthermore, since B already knew that the other odd number was different from b, we must have $b \geq 7$.

The constraints $b \in \{7, 9, 11\}$, $a + b + c = 14$ and a being odd, yield the following possible triples (a, b, c) after A and B have spoken:

$$(1, 7, 6), (3, 7, 4), (5, 7, 2), (1, 9, 4), (3, 9, 2), (1, 11, 2).$$

Now, if $c = 4$ or $c = 2$, then C would not know a and b for certain. Hence we conclude that $(a, b, c) = (1, 7, 6)$.

Bonus: Again, a is odd. We cannot have $b \in \{7, 9, 11\}$ or else B would not have needed to wait for A to speak. We cannot have $b \in \{1, 3, 5\}$ or else B would not have known that $a \neq b$.

If $b \in \{4, 8, 12\}$, then $a + c = 14 - b$ is two times an odd number. So B would not have known that $a \neq c$. This leaves the possibilities $b \in \{2, 6, 10\}$. Together with $a + b + c = 14$ and a being odd, we have the following possibilities for (a, b, c) after A and B have spoken:

$$(1, 2, 11), (3, 2, 9), (5, 2, 7), (7, 2, 5), (9, 2, 3), (11, 2, 1),$$
$$(1, 6, 7), (3, 6, 5), (5, 6, 3), (7, 6, 1),$$
$$(1, 10, 3), (3, 10, 1).$$

Again we disregard triples where c is not unique, and conclude that there are two possibilities: $(a, b, c) = (1, 2, 11)$ or $(a, b, c) = (3, 2, 9)$. We cannot know which one is correct for sure, and neither can the second apprentice!

Serious summation

Solution by Samuel Mueller: Rationalising the denominators and using a few simple identities:

$$\sum_{n=1}^{9800} \frac{1}{\sqrt{n + \sqrt{n^2 - 1}}} = \sum_{n=1}^{9800} \sqrt{\frac{1}{n + \sqrt{n^2 - 1}} \times \frac{n - \sqrt{n^2 - 1}}{n - \sqrt{n^2 - 1}}}$$
$$= \sum_{n=1}^{9800} \sqrt{n - \sqrt{n^2 - 1}}$$
$$= \frac{1}{\sqrt{2}} \sum_{n=1}^{9800} \sqrt{2n - 2\sqrt{(n+1)(n-1)}}$$
$$= \frac{1}{\sqrt{2}} \sum_{n=1}^{9800} \sqrt{\left(\sqrt{n+1} - \sqrt{n-1}\right)^2}$$
$$= \frac{1}{\sqrt{2}} \sum_{n=1}^{9800} \left(\sqrt{n+1} - \sqrt{n-1}\right).$$

This is a telescoping sum. Most of the terms cancel out as follows,

$$\frac{1}{\sqrt{2}} \left(\sqrt{9801} - \sqrt{9799} + \sqrt{9800} - \sqrt{9798} + \cdots + \sqrt{3} - \sqrt{1} + \sqrt{2} - \sqrt{0}\right)$$
$$= \frac{1}{\sqrt{2}} \left(\sqrt{9801} + \sqrt{9800} - \sqrt{1} - \sqrt{0}\right)$$
$$= \frac{1}{\sqrt{2}} \left(99 + 70\sqrt{2} - 1\right)$$
$$= 70 + 49\sqrt{2}.$$

Reality television

Solution by Christopher Ryba: Let the set of contestants be $A = \{a_1, \ldots, a_n\}$. For each contestant a_i, define $f(a_i)$ to be the number of contestants in his or her team, including a_i. Also define $g(a_i)$ to be the number of contestants in his or her room,

including a_i as well. Now

$$\sum_{a_i \in A} \frac{1}{f(a_i)} = \text{the number of teams} = 11$$

$$\sum_{a_i \in A} \frac{1}{g(a_i)} = \text{the number of rooms} = 8.$$

Hence

$$\sum_{a_i \in A} \left(\frac{1}{f(a_i)} - \frac{1}{g(a_i)} \right) = 3.$$

Since the functions f and g only return positive integers, for each a_i we must have $|1/f(a_i) - 1/g(a_i)| \leq 1$. So for the above sum to be 3,

$$\frac{1}{f(a_i)} - \frac{1}{g(a_i)} > 0$$

must hold for tjree choices of a_i. This rearranges to $g(a_i) > f(a_i)$. Therefore there exist three contestants with more roommates than teammates.

Scissors and shapes

1. *Solution by Alan Jones:* First note that for any given right-angled triangle, the cut is unique and well defined. Assume Edward can reach a state with no congruent triangles. At least three out of the four original triangles must be cut at least once. This results in three small congruent triangles labelled S and three large congruent triangles labelled L, as shown in the diagram below.
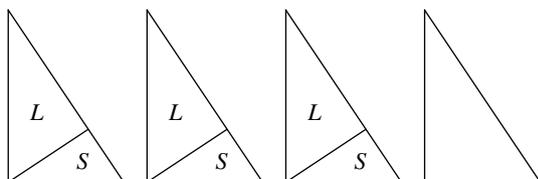
Once again, for there to be no congruent triangles, at least two of the S triangles must be cut. The same goes for at least two of the L triangles. This creates two SL and two LS triangles, as shown below.
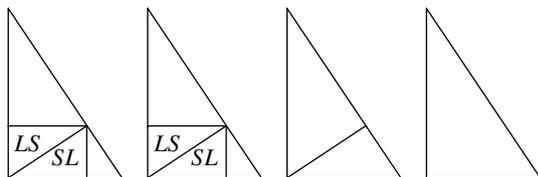
However, the SL and LS triangles are congruent. So Edward has returned to the original state of four congruent triangles (not to mention any other possibly congruent triangles out of the remaining pieces), despite only performing necessary cuts. Therefore it is not possible for Edward to create pairwise incongruent triangles using a finite number of cuts.

2. *Solution by Ross Atkins:* Notice that after each slice-flip-glue move, the area and perimeter of the shape remains constant. So it is sufficient to prove that no triangle has the same area and perimeter as a square.

Without the loss of generality, let the square be a unit square, and let (a, b, c) be the side lengths of a triangle of perimeter $a + b + c = 4$. By Heron's formula, the area of such a triangle would be

$$\sqrt{2(2-a)(2-b)(2-c)}.$$

Now by the arithmetic and geometric means (AM-GM) inequality

$$\frac{2 + (2-a) + (2-b) + (2-c)}{4} \geq \sqrt[4]{2(2-a)(2-b)(2-c)}$$

$$\implies \qquad\qquad\qquad 1 \geq \sqrt{2(2-a)(2-b)(2-c)}$$

$$\implies \qquad \text{Area of the square} \geq \text{Area of the triangle}.$$

In order to achieve equality in the AM-GM inequality, we require $2 = 2-a = 2-b = 2-c$ which cannot hold. So the triangle cannot have the same area as the square.

Therefore Edward can never transform a square into a triangle using these moves.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.