Puzzle corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner No. 20. Each issue will include a handful of fun, yet intriguing, puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes SA 5095.

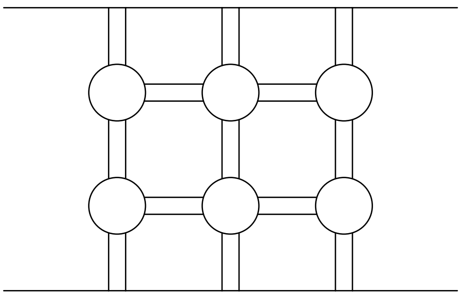
The deadline for submission of solutions for Puzzle Corner 20 is 1 January 2011. The solutions to Puzzle Corner 20 will appear in Puzzle Corner 22 in the May 2011 issue of the *Gazette*.

Lousy labelling

Three boxes are on the table. One has red balls, one has blue balls, and one has balls of both colours. Three labels are made for the boxes, but they are misplaced so none of the boxes are labelled correctly. How many balls would you need to retrieve from the boxes in order to determine the correct labelling?

Broken bridges

There are thirteen bridges connecting the banks of River Pluvia and its six piers, as shown in the diagram below:



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On an extremely stormy night, each bridge has a 50% chance of being damaged by the rainfall. What is the probability that the locals can still cross the river using undamaged bridges the next morning?

Trick question

Find all real solutions to the equation

$$\sqrt{x+4\sqrt{x-4}}-\sqrt{x-4\sqrt{x-4}}=4.$$

Photo: Jack Horst

Clock shop

A clock shop has 10 accurate clocks of various sizes on display. Prove that there exists a moment in time when the sum of all pairwise distances between the tips of the minute hands is greater than the sum of all pairwise distances between the centres of the clocks.


Photo: Kamila Turton

Super knight tour

In a game of super chess, a super knight can move diagonally across a 4×3 rectangle (as opposed to a standard knight which moves diagonally across a 3×2 rectangle). Can the super knight perform a knight tour on a 12×12 super chessboard, i.e. use a sequence of moves to visit every square exactly once?

Consecutive sums

(1) What is the smallest number that can be expressed as a sum of consecutive positive integers in exactly 2010 different ways? Note that a sum must contain at least two summands.

(2) Can you find a number which can be expressed as a sum of an even number of consecutive positive integers in exactly 2010 different ways? Can you find one that is smaller than the answer to part (1)?

Solutions to Puzzle Corner 18

The $50 book voucher for the best submission to Puzzle Corner 18 is awarded to David Butler. Congratulations!

Page numbers

Solution by: Laura McCormick

The first page that Tom tore out was odd. Since it is impossible to tear out only one side of a leaf, the final removed page must be even. Since the final page is comprised of the same digits as the first, it must be either 138 or 318. But the last page cannot precede the first! So the final page must be 318, and a total of 136 pages (or 68 leaves) were torn out.

Fraction practice

Solution by: David Butler

We will prove the general case: if the starting list of fractions is

$$\frac{1}{a} \qquad \frac{1}{a+1} \qquad \cdots \qquad \frac{1}{a+n},$$

then the final fraction would be $((a-1)!\,n!)/(a+n)!$.

This result is established by induction on n for all a. The base case of $n=1$ can be easily checked. Assume the result for $n=k$ and consider the case of $n=k+1$. The starting list is

$$\frac{1}{a} \qquad \frac{1}{a+1} \qquad \cdots \qquad \frac{1}{a+k} \qquad \frac{1}{a+k+1}.$$

The second last step of the game has two fractions in the list. The left-hand fraction is produced by playing the game beginning with the left-hand k fractions from the original list. By our assumption this gives $((a-1)!\,k!)/(a+k)!$.

The right-hand fraction is produced by playing the game beginning with the right-hand k fractions from the original list. We can once again use our assumption, after replacing a by $a+1$, and obtain $(a!\,k!)/(a+1+k)!$.

So, the final fraction is

$$\begin{aligned}
\frac{(a-1)!\,k!}{(a+k)!} - \frac{a!\,k!}{(a+1+k)!} &= \frac{(a-1)!\,k!\,(a+1+k)}{(a+k)!\,(a+1+k)} - \frac{a!\,k!}{(a+1+k)!} \\
&= \frac{(a-1)!\,k!\,((a+1+k)-a)}{(a+k+1)!} \\
&= \frac{(a-1)!\,k!\,(k+1)}{(a+k+1)!} \\
&= \frac{(a-1)!\,(k+1)!}{(a+k+1)!}
\end{aligned}$$

and the induction is complete.

Applying to Franny's problem, using $a = 1$ and $n = 99$, the final fraction must be $1/100$.

Invisible point

Solution by: Randell Heyman

With only two lines, any responses from the elf will always leave us with an infinite amount of space, some of which has to be outside of the square. Hence two lines are not enough to verify if the point is inside the square.

However, it is possible to determine the location of the point using three lines. Choose the first two lines to be the diagonals of the square. The elf's responses will provide us a quadrant containing the point. The last line should be the side of the square in that quadrant and the elf's response will confirm whether the point is inside the square. The same process can also handle the 'on the line' responses by the elf.

Differing views

Solution by: Pratik Poddar

Yes they can both be right and the longest such sequence has 11 terms.

Suppose a length 12 sequence exists, let it be $\{a_1, a_2, \ldots, a_{12}\}$. Consider the following array of numbers.

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} \\ a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{matrix}$$

According to the optimist, the sum of each row is positive. But according to the pessimist, the sum of each column is negative. The sum of the array is therefore both positive and negative at the same time, a contradiction.

There are many possible constructions of a length 11 sequence; here is an example:

$$\{1, -1.6, 1, 1, -1.6, 1, -1.6, 1, 1, -1.6, 1\}.$$

Coin conundrum

Solution by: Ross Atkins

The answer is yes, it is always possible to flip every coin.

First let us define a *hyper-flip*. A hyper-flip on coin X is a combination of moves that flips every coin in the arrangement except coin X.

This is a proof by induction. Assume that any arrangement of $n - 1$ coins can be flipped using an appropriate sequence of moves. This means that for any subset of size $n-1$ of our n coins, there exists a sequence of moves that flips this subset. This sequence might flip the n^{th} coin or it might not. If the sequence of moves does flip the nth coin (for any one of the $(n-1)$-coin subsets), then we are done. It suffices to consider the cases where it's possible to perform a hyper-flip about any arbitrary coin.

If n is even, we simply perform a hyper-flip on each coin successively. As a result, each coin is flipped $n-1$ times.

If n is odd, then at least one coin is touching an even number of other coins. This is because the total of adjacencies summed over all coins, which equals to twice the total number of adjacent pairs of coins, is even. Let X be a coin with $2k$ neighbours. Perform a hyper-flip on X, plus a hyper-flip on each of the neighbours of X. Then finish with an ordinary flip on X. Each coin was flipped exactly $2k+1$ times.

In both cases, every coin was flipped an odd number of times, hence from head to tail. This completes the proof.

Rational points

Solution by: Norman Do

(1) Suppose that such a polygon exists and let the vectors corresponding to its sides — traversed in an anticlockwise manner — be (a_i, b_i) for $i = 1, 2, \dots, 1001$. By performing a dilation, we can arrange it so that the coordinates are not only rational, but integral. In fact, we can do this in such a way that the integers $a_1, a_2, \dots a_{1001}, b_1, b_2, \dots, b_{1001}$ have no common factor. In particular, this means that one of these numbers must be odd and we may assume without loss of generality that it is a_1.
The fact that all sides are equal in length implies that there is some integer d such that $a_i^2 + b_i^2 = d$ for each $i = 1, 2, \dots, 1001$. The fact that the polygon is closed implies that $\sum a_i = \sum b_i = 0$. Using this in conjunction with the fact that $x^2 \equiv x \pmod 2$ allows us to deduce that $\sum a_i^2 \equiv \sum b_i^2 \equiv 0 \pmod 2$. Therefore, we have

$$1001d = \sum a_i^2 + \sum b_i^2 \equiv 0 \pmod 2 \quad \Rightarrow \quad d \equiv 0 \pmod 2.$$

Since $a_1^2 + b_1^2 = d$ is even and we have assumed that a_1 is odd, it follows that b_1 is odd. But the square of an odd number is always congruent to 1 modulo 4, so $d = a_1^2 + b_1^2 \equiv 2 \pmod 4$. In fact, the only way that $a^2 + b^2 \equiv 2 \pmod 4$ is if both a and b are odd. It now follows that all of the numbers $a_1, a_2, \dots, a_{1001}$ are odd, which contradicts the fact that $a_1 + a_2 + \cdots + a_{1001} = 0$. So we conclude that no such polygon exists.

(2) Consider the points $P_m = (m, m^2)$, where $m = 1, 2, \dots, 1001$. No three such points are collinear as they all lie on the parabola $y = x^2$. Now, the distance between the points P_m and P_n is

$$\sqrt{(m-n)^2 + (m^2 - n^2)^2} = (m-n)\sqrt{(m+n)^2 + 1}.$$

For this to be a rational number, we would need $(m+n)^2+1$ to be a perfect square. However, the only perfect squares which differ by one are 0 and 1. This forces $m+n=0$ which is clearly impossible. Therefore, the distance between any pair of points is irrational.
On the other hand, all of the points $P_1, P_2, \ldots, P_{1001}$ are lattice points—that is, they have integer coordinates. Pick's theorem states that the area of any polygon whose vertices are lattice points has area $I + \frac{B}{2} - 1$, where I is the number of lattice points interior to the polygon and B is the number of lattice points on the boundary of the polygon. It follows that the area of any triangle formed by any triple is rational.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.