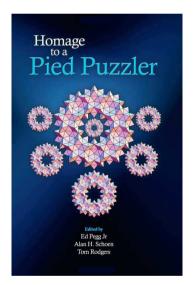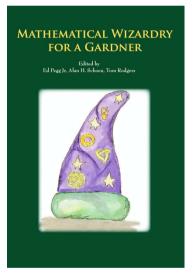**Homage to a Pied Puzzler**
and
Mathematical Wizardry for a Gardner

Ed Pegg Jr, Alan H. Schoen and Tom Rodgers (eds)
A K Peters Ltd, 2009, ISBN: 978-1-56881-315-8 and 978-1-56881-447-6

Since the retirement of Martin Gardner from *Scientific American* his admirer Tom Rodgers has organised a three-day meeting every second year. Rodgers edited a book commemorating Gardner's ninetieth birthday; this book was reviewed in the *Gazette* by Phill Schultz in March 2010. Gardner died two months later in May at the age of 95. These books, henceforth *HPP* and *MWG*, contain versions of selected talks at the 2006 meeting. Rodgers is one co-editor of these volumes. Ed Pegg of Wolfram Research and webmaster of mathpuzzle.com is another.

Four of Gardner's *Scientific American* columns have now become firmly established in the mathematical consciousness. The first described the RSA encryption process 'which would take millions of years to break'. This is alive and well in making communications secure today. The British Government Communications Headquarters (corresponding to the American National Security Agency and the Australian Defence Signals Directorate) has now released on its website, earlier internal documents on this subject. These were considered to be very dangerous secrets at the time. Gardner let that cat out of the bag. The second concerned John Conway's *Game of Life* and set the cellular automaton up as a fertile field for analysis. The third classic was Penrose tilings, launched with Conway's assistance in 1971 and nowadays an interesting and perhaps pivotal example of a non-commutative geometry in the sense of Alain Connes. The column in *Scientific*

American also popularised the mathematical art of Maurits Escher among the mathematical community.

Such meetings, or perhaps a hypothetical journal named *Mathematical Curiosities*, cannot be expected to match the significance of RSA encryption, the Game of Life, Penrose tilings or the art of Escher very often. But there must be a future for publishing high quality material of this type. But time has moved on since the Gardner era, with the web emerging as the means to distribute esoteric information. A lovely example of this may be found by asking Google Video for *longest mates*. This yields a chess position with $K + Q + N$ versus $K + R + B + N$ where with best play on both sides the $K + Q + N$ win after 524 moves (each). This is the longest road to checkmate for the seven piece endgame and was found by Marc Bourzutschky and Yakov Konoval by an exhaustive search using special ingenious programming techniques on inexpensive standard computing facilities. The PhD thesis of Lewis Stiller is important for programs of this sort. Gardner could and did describe such innovations so well. Although Google Video presents the 524 moves with an appropriate musical background, it seems a pity that the development has not been written up on paper.

Computer graphics are another post-Gardner phenomenon. One can but admire the pictures produced by Sándor Kabai (HPP 53–73). His article begins:

> Seven-fold symmetry is explored with the help of graphs generated by the software Mathematica. ... It is shown how these rings are related to the Yoshimura shape and to a possible spacecraft design.

Continuing with the number seven, the distinguished veteran Michael Longuet-Higgins describes (HPP 85–91) organic material with a seven-fold symmetry that can be found in the oceans. This is (very) loosely related to the five-fold symmetry in quasi-crystals. And yes, Akio Hizume in *Quasi-Periodic Essays in Architectural and Musical Form* (MWG 211–219) does manage to bring in some of the work of Roger Penrose.

Neil Sloane of ATT Research has been collecting integer sequences for around 40 years, becoming extremely well known for his online integer sequence facility. This format has long since replaced his 1973 book, confirming once again that we are living in a new age with new technology. Sloane's choice (HPP 93–110) for the most interesting and still unresolved sequence is the *Kissing Number Problem*: given n determine the maximum number of unit balls in the n-dimensional Euclidean space that can touch another such ball. Even with $n = 4$ this is quite non-trivial, and only recently solved by O. R. Musin.

Yossi Elran in *Retrolife* (HPP 129–135) discusses the question of determining what a configuration in *Life* came from. There will not be unique answers but something can be said. Google and Google Scholar with input 'Conway Life' come up with much information as well as freebie software.

David Lister has written about *Martin Gardner and Paperfolding* (MWG 9–27) and thus set out what was a major theme in Gardner's work. In so doing he provides a useful introduction to the whole Gardner movement for the younger generation.

Martin Gardner was fond of publishing selected puzzles. This *Gazette* now makes a regular feature of such. Not bad, Norman, and now Ivan: keep it up! Some puzzle material is also to be found in these books.

Those who are more than skeptical about magic (meaning card tricks and the like) can learn from these books how it can be produced by error-correcting codes.

It emerges that the celebrated American puzzle maker Sam Loyd did not invent the 15-16 puzzle but Serhiy Grabarchuk in *The Two Ovals-to-Table Story* (HPP 175–179) is able to go from one of Loyd's classic problems to show what dissection theory is now all about. Was Loyd the first to produce the short autobiography 'I was born of rich but honest parents'?

The optimal outcome from these books would be the development of a viable publication outlet for the best mathematical curiosities. The culture is more than productive enough to generate the required material. Holding regular Gardner memorial meetings is one way of achieving this aim.

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Structure and Randomness

In 2007, not long after he was awarded the Fields Medal, Terence Tao started a mathematical research blog. This was a natural outgrowth of his webpage, where he had for several years posted 'short stories' and other articles describing in informal terms important recent mathematical developments, due to himself or others. It has become probably the most widely read mathematical blog of its type (Gowers, Boucherds and others also run blogs aimed at research mathematicians). Interest in the articles on this blog has been such that the American Mathematical Society has begun publishing compilations of articles from it, at the planned rate of one book per year. This article is a review of the first book, consisting of articles posted in the year 2007.

This is a unique book (except for its sequel, *Poincaré's Legacies: Pages from Year Two of a Mathematical Blog* by the same author!) in several respects. Firstly, its structure is unique. I am not aware of any other mathematics book written by a single author that is a compilation of articles on a large range of different mathematical topics, and addressed primarily to research mathematicians. (Indeed there are only a handful of contemporary mathematicians who would be capable of writing such a book.) Secondly, the book addresses the question 'How to do mathematics', but at a higher level (aimed at practising research mathematicians)

than anywhere else I can think of. It is like 'How to solve it' for professional mathematicians. Thirdly, the quality of exposition is mostly superb. Indeed, a number of the articles in this collection could be used as models of clear mathematical writing.

To give an idea of the scope of the book, here are the names of some of the articles: 'Compressed sensing and single-pixel cameras', a current hot topic in applied mathematics; 'The Lebesgue differentiation theorem and the Szemerédi regularity lemma'; 'Ultrafilters, non-standard analysis, and epsilon management', which explains the way in which non-standard analysis could be used to simplify analysis arguments involving several interdependent epsilons; 'Dyadic models' (harmonic analysis); 'Nonfirstorderisability', about the surprising fact that some quite reasonable mathematical statements cannot be expressed in first-order logic; 'The crossing number inequality' (in graph theory); 'Hilbert's Nullstellensatz'; 'Mahler's conjecture for convex bodies'; 'The parity problem in sieve theory'. Only a few of these are expositions of his own research. There are also sections consisting of the text of lectures that Tao has given, including one, 'Structure and Randomness', that gives this book its title and is an elaboration of his Fields address.

It would be tedious to try to review each article individually, but I will comment on a few of them. 'Quantum mechanics and Tomb Raider' is a delightfully playful article which explains certain paradoxical elements of quantum mechanics through a computer game analogy. In this analogy, the game's characters reside in an internal universe that can be modified by the player through 'save' and 'restore' functions. These functions model the way in which a quantum-mechanical wave function collapses when we make a measurement, say of the energy of a particle. Tao then shows how two standard paradoxes in quantum mechanics, the double-slit experiment and Bell's inequality violation, have analogues in the internal universe of the characters in the game. The article is highly entertaining as well as thought-provoking.

In the article 'Soft analysis, hard analysis and the finite convergence principle', he gives a detailed discussion of the differences between infinitary and finitary mathematics, illustrated by the example of the familiar theorem that a bounded monotone sequence of real numbers converges. This is an infinitary statement because it asserts convergence, but gives no bounds on the *rate* at which the sequence converges. Tao shows how to obtain a finitary version of this statement that is logically equivalent, in the sense that each version can be proved from the other. The surprising thing is that the form of this finitary statement is not at all obvious. Here it is:

Theorem 1 (Finite convergence principle). *Let $\epsilon > 0$ and $F\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ be given. Then there exists a natural number M depending on ϵ and F, such that for all monotone sequences $0 \le x_1 \le x_2 \le \cdots \le x_M \le 1$, there exists N with $N < N + F(N) \le M$ such that $|x_n - x_m| < \epsilon$ whenever $N \le n, m \le N + F(N)$.*

It's a rather bizarre statement at first glance, but Tao argues that this is the correct finitary version, by showing that each implies the other. His exposition for how to obtain the theorem above is masterly; he takes us through the whole thought process, deriving a number of incorrect but 'increasingly correct' versions of this result. In this sense, Tao is teaching us how to think about mathematics, and how a philosophical approach to mathematics can be not just intellectually satisfying or amusing, but actually useful. (Finitary versions of theorems in arithmetic combinatorics, such as Szemerédi's theorem, were crucial in Green and Tao's celebrated theorem on arithmetic progressions in the prime numbers.)

My favourite article in the whole collection is 'Amplification, arbitrage, and the tensor power trick', in which he discusses how certain inequalities can be improved, if there is a symmetry present on one side of the equation that is not present in the other. The discussion starts with a very elementary example, that of proving the Cauchy–Schwarz inequality from the trivial inequality

$$\operatorname{Re} x \cdot y \le \tfrac{1}{2}(|x|^2 + |y|^2), \qquad x, y \in \mathbb{C}^n \tag{1}$$

(equivalent to $|x - y|^2 \ge 0$). This is weaker than the Cauchy–Schwarz inequality

$$|x \cdot y| \le |x||y|; \tag{2}$$

in fact, the left-hand side of (1) is smaller than that of (2), while the right-hand side is bigger. Note that the LHS and RHS of (1) have different symmetries: the RHS is invariant under a phase rotation $x \to xe^{i\theta}$, θ real, the LHS is not; while the LHS is invariant under a dilation $x \to \lambda x$, $y \to \lambda^{-1} y$, while the RHS is not. Using the first invariance, we may choose θ so that $\operatorname{Re}(e^{i\theta}x) \cdot y = |x \cdot y|$, allowing us to replace the LHS of (1) with that of (2). Using the second invariance, we can choose λ to maximise the RHS and it is a simple computation to show that this happens when $\lambda = \sqrt{|y|/|x|}$, in which case the RHS is replaced by the RHS of (2). We can think of these manipulations as sharing or 'arbitraging' some of the symmetry on one side of the equation with the other side. The strengthening in the inequality thus obtained is the 'amplification'.

This discussion can be easily understood by a bright undergraduate. But the article goes on to show how this 'arbitraging' trick can be used in much more sophisticated contexts, e.g. to obtain vector-valued (Banach space) inequalities from scalar-valued inequalities, or some sophisticated inequalities in arithmetic combinatorics. I found this article extremely useful in crystallising some ideas that I already understood but only vaguely or superficially. I suspect that this understanding will be useful in my research at some point in the future.

The finitary/infinitary dichotomy is one theme explored in a number of articles in this collection. Another dichotomy is the one in the title of the book, structure versus randomness. The 'Simons lecture series: structure and randomness'

is a philosophical article about how structure and randomness manifest themselves in a number of different mathematical areas, such as additive number theory (e.g. counting primes or twin primes), ergodic theory, graph theory, or PDE theory. In his words:

> Hybrid sets — [are] sets which exhibit some features of structure and some features of pseudorandomness .... In order to study hybrid objects, one needs a large variety of tools: one needs tools such as algebra and geometry to understand the structured component, one needs tools such as analysis and probability to understand the pseudorandom component, and one needs tools such as decompositions, algorithms, and evolution equations to separate the structure from the pseudorandomness.

This very general statement is then fleshed out with a number of examples, including several pertinent to Tao's proof with Green of the existence of arbitrarily long arithmetic progressions in the primes.

The article 'Why global regularity for Navier–Stokes is hard' is a good example of Tao using his blog to disseminate folklore-type information that is well known in the relevant community of mathematical experts, but rarely written down (and thus difficult for novices to learn). In this case, it is a clear explanation about why supercritical PDEs such as the Navier–Stokes equations are difficult to solve, particularly for long time. Tao gives a list of possible approaches to Navier–Stokes and summarises the essential problems with each one.

The article 'The parity problem in sieve theory' is a beautifully clear introduction to the general idea of using sieving to count sets of integers, such as prime numbers, twin primes, et cetera, in (say) the interval $[N, 2N]$, for large N. Along the way, Tao gives a clear and elegant description of 'smooth' sets of integers and how it is natural to consider p-adic metrics when doing this. He explains why sieve theory can successfully give reasonably good upper bounds on, say, the number of twin primes, but not good lower bounds, so that it remains an extremely difficult problem to show that there are infinitely many twin primes.

This book is a marvellous addition to any mathematician's bookshelf. The articles are of exceptional interest, and the quality of exposition varies from very good to outstanding. Overall, it is a rare privilege to be able to eavesdrop on the thought process of one of the greatest living mathematicians who, fortunately for the rest of us, also has a strong interest in communicating his insights to the global community of research mathematicians.

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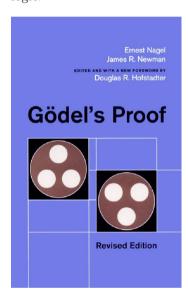Department of Mathematics, Australian National University, Canberra, ACT 0200.
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Gödel's Proof (Revised Edition)

Ernest Nagel and James R. Newman
Edited and with a new foreword by Douglas R. Hofstadter
New York University Press, 2001, ISBN: 978-0-8147-5837-3
Re-issued by New York University Press, 2008

Most mathematics students have heard of Gödel's theorems and have a vague idea that they show that all mathematics is flawed or uncertain. This little book is an excellent and easy place to start if you want to understand what Gödel's theorems and proofs are about, and get a firm footing to delve deeper into mathematical logic.

The book was first published in 1958, and this current paperback edition (put out in 2001) looks much friendlier, with its lavender blue cover, than the old yellowing copies in the university library. The 'new foreword' should be skipped unless you are a fan of Hofstadter (author of [1]) and want to hear his personal stories of reading the book, meeting Nagel and so forth. I would recommend getting straight into the original text.

The book does not pretend to be a complete treatise on the proofs, but is meant to motivate and set the scene for them, and encourage you to learn more of them from other sources. While researching for this review I read the review in the AMS *Notices* by philosopher Timothy McCarthy [3], which specifies some of the places the text has issues. As a non-expert in logic, I do not appreciate all of the finer points. At the beginning of his review, McCarthy [3] quotes from Hardy's *A Mathematician's Apology*: 'no book is wholly bad that fires the imagination of clever young people'. The book reviewed here certainly does fire the imagination, even of the not-so-clever or not-so-young like this reviewer.

Chapter 7 of the book begins:

> Gödel's paper is difficult. Forty-six preliminary definitions, together with several important preliminary propositions, must be mastered before the main results are reached. We shall take a much easier road; nevertheless, it should afford the reader glimpses of the ascent and of the crowning structure.

and this, I believe, it does.

One thing I especially like is the detailed description of the Gödel numbering trick. This appeals to mathematicians who like to see how things work under the bonnet. In other places I have seen the numbering skipped completely, saying

that its precise definition is not important. In this text, the authors compute the number corresponding to the statement '0 = 0' to be 243 million, which impressed me quite a bit.

There are several other books that discuss and describe Gödel's work to the lay person. An interesting one I found recently is [2], which looks into the many misstatements and misuses of what Gödel's theorems say. For example:

> As Gödel demonstrated, all consistent formal systems are incomplete, and all complete formal systems are inconsistent. The U.S. Constitution is a formal system, after a fashion. The Founders made the choice of incompleteness over inconsistency, and the Judicial Branch exists to close that gap of incompleteness.

In summary, I encourage everyone who hasn't read *Gödel's Proof* to get hold of a copy — it is an easy read for such powerful contents. It is also instructive to show how one can write and communicate complex ideas in a friendly and accessible way, without dumbing down the content.

Note from the publisher: In Australia this new edition is available through Footprint Books.

References

[1] Hofstadter, D.R. (1999). *Gödel, Escher, Bach: An Eternal Golden Braid*. Basic Books, Inc., New York, USA.
[2] Franzén, T. (2005). *Gödel's Theorem: An Incomplete Guide to its Use and Abuse*. A K Peters, Ltd, Wellesley, MA, USA.
[3] McCarthy, T. (2004). Book Review: *Gödel's Proof. Notices of the AMS* **51,** 333–337.

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